CHAPTER 2

Second-Order Linear ODEs

Many important applications in mechanical and electrical engineering, as shown in Secs. 2.4, 2.8, and 2.9, are modeled by linear ordinary differential equations (linear ODEs) of the second order. Their theory is representative of all linear ODEs as is seen when compared to linear ODEs of third and higher order, respectively. However, the solution formulas for second-order linear ODEs are simpler than those of higher order, so it is a natural progression to study ODEs of second order first in this chapter and then of higher order in Chap. 3.

Although ordinary differential equations (ODEs) can be grouped into linear and nonlinear ODEs, nonlinear ODEs are difficult to solve in contrast to linear ODEs for which many beautiful standard methods exist.

Chapter 2 includes the derivation of general and particular solutions, the latter in connection with initial value problems.

For those interested in solution methods for Legendre's, Bessel's, and the hypergeometric equations consult Chap. 5 and for Sturm–Liouville problems Chap. 11.

COMMENT. ***Numerics for second-order ODEs can be studied immediately after this chapter.*** See Sec. 21.3, which is independent of other sections in Chaps. 19–21.

Prerequisite: Chap. 1, in particular, Sec. 1.5.
Sections that may be omitted in a shorter course: 2.3, 2.9, 2.10.
References and Answers to Problems: App. 1 Part A, and App. 2.

2.1 Homogeneous Linear ODEs of Second Order

We have already considered first-order linear ODEs (Sec. 1.5) and shall now define and discuss linear ODEs of second order. These equations have important engineering applications, especially in connection with mechanical and electrical vibrations (Secs. 2.4, 2.8, 2.9) as well as in wave motion, heat conduction, and other parts of physics, as we shall see in Chap. 12.

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

and **nonlinear** if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in y and its derivatives,* whereas the functions p, q, and r on the right may be any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term.

The definitions of homogeneous and nonhomogenous second-order linear ODEs are very similar to those of first-order ODEs discussed in Sec. 1.5. Indeed, if $r(x) \equiv 0$ (that is, $r(x) = 0$ for all x considered; read "$r(x)$ is identically zero"), then (1) reduces to

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and is called **homogeneous**. If $r(x) \not\equiv 0$, then (1) is called **nonhomogeneous**. This is similar to Sec. 1.5.

An example of a nonhomogeneous linear ODE is

$$y'' + 25y = e^{-x} \cos x,$$

and a homogeneous linear ODE is

$$xy'' + y' + xy = 0, \qquad \text{written in standard form} \qquad y'' + \frac{1}{x}y' + y = 0.$$

Finally, an example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$

The functions p and q in (1) and (2) are called the **coefficients** of the ODEs.

Solutions are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$

is called a *solution* of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown y by h, the derivative y' by h', and the second derivative y'' by h''. Examples are given below.

Homogeneous Linear ODEs: Superposition Principle

Sections 2.1–2.6 will be devoted to **homogeneous** linear ODEs (2) and the remaining sections of the chapter to nonhomogeneous linear ODEs.

Linear ODEs have a rich solution structure. For the homogeneous equation the backbone of this structure is the *superposition principle* or *linearity principle,* which says that we can obtain further solutions from given ones by adding them or by multiplying them with any constants. Of course, this is a great advantage of homogeneous linear ODEs. Let us first discuss an example.

EXAMPLE 1 **Homogeneous Linear ODEs: Superposition of Solutions**

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all x. We verify this by differentiation and substitution. We obtain $(\cos x)'' = -\cos x$; hence

$$y'' + y = (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$

Similarly for $y = \sin x$ (verify!). We can go an important step further. We multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by, say, -2, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x) = -4.7 \cos x + 2 \sin x + 4.7 \cos x - 2 \sin x = 0. \quad \blacksquare$$

In this example we have obtained from $y_1 (= \cos x)$ and $y_2 (= \sin x)$ a function of the form

$$y = c_1 y_1 + c_2 y_2 \qquad (c_1, c_2 \text{ arbitrary constants}). \tag{3}$$

This is called a **linear combination** of y_1 and y_2. In terms of this concept we can now formulate the result suggested by our example, often called the **superposition principle** or **linearity principle**.

THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

PROOF Let y_1 and y_2 be solutions of (2) on I. Then by substituting $y = c_1 y_1 + c_2 y_2$ and its derivatives into (2), and using the familiar rule $(c_1 y_1 + c_2 y_2)' = c_1 y_1' + c_2 y_2'$, etc., we get

$$\begin{aligned} y'' + py' + qy &= (c_1 y_1 + c_2 y_2)'' + p(c_1 y_1 + c_2 y_2)' + q(c_1 y_1 + c_2 y_2) \\ &= c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0, \end{aligned}$$

since in the last line, $(\cdots) = 0$ because y_1 and y_2 are solutions, by assumption. This shows that y is a solution of (2) on I. $\blacksquare$

CAUTION! Don't forget that this highly important theorem holds for *homogeneous linear* ODEs only but ***does not hold*** for nonhomogeneous linear or nonlinear ODEs, as the following two examples illustrate.

EXAMPLE 2 **A Nonhomogeneous Linear ODE**

Verify by substitution that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear ODE

$$y'' + y = 1,$$

but their sum is not a solution. Neither is, for instance, $2(1 + \cos x)$ or $5(1 + \sin x)$. $\blacksquare$

EXAMPLE 3 **A Nonlinear ODE**

Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy' = 0,$$

but their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by -1! $\blacksquare$

Initial Value Problem. Basis. General Solution

Recall from Chap. 1 that for a first-order ODE, an *initial value problem* consists of the ODE and one *initial condition* $y(x_0) = y_0$. The initial condition is used to determine the *arbitrary constant* c in the *general solution* of the ODE. This results in a unique solution, as we need it in most applications. That solution is called a *particular solution* of the ODE. These ideas extend to second-order ODEs as follows.

For a second-order homogeneous linear ODE (2) an **initial value problem** consists of (2) and two **initial conditions**

$$y(x_0) = K_0, \qquad y'(x_0) = K_1. \tag{4}$$

These conditions prescribe given values K_0 and K_1 of the solution and its first derivative (the slope of its curve) at the same given $x = x_0$ in the open interval considered.

The conditions (4) are used to determine the two arbitrary constants c_1 and c_2 in a **general solution**

$$y = c_1 y_1 + c_2 y_2 \tag{5}$$

of the ODE; here, y_1 and y_2 are suitable solutions of the ODE, with "suitable" to be explained after the next example. This results in a unique solution, passing through the point (x_0, K_0) with K_1 as the tangent direction (the slope) at that point. That solution is called a **particular solution** of the ODE (2).

EXAMPLE 4 **Initial Value Problem**

Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5.$$

Solution. ***Step 1. General solution.*** The functions $\cos x$ and $\sin x$ are solutions of the ODE (by Example 1), and we take

$$y = c_1 \cos x + c_2 \sin x.$$

This will turn out to be a general solution as defined below.

Step 2. Particular solution. We need the derivative $y' = -c_1 \sin x + c_2 \cos x$. From this and the initial values we obtain, since $\cos 0 = 1$ and $\sin 0 = 0$,

$$y(0) = c_1 = 3.0 \qquad \text{and} \qquad y'(0) = c_2 = -0.5.$$

This gives as the solution of our initial value problem the particular solution

$$y = 3.0 \cos x - 0.5 \sin x.$$

Figure 29 shows that at $x = 0$ it has the value 3.0 and the slope -0.5, so that its tangent intersects the x-axis at $x = 3.0/0.5 = 6.0$. (The scales on the axes differ!) ■

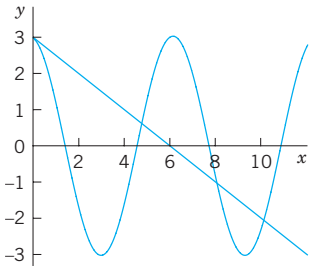

Fig. 29. Particular solution and initial tangent in Example 4

Observation. Our choice of y_1 and y_2 was general enough to satisfy both initial conditions. Now let us take instead two proportional solutions $y_1 = \cos x$ and $y_2 = k \cos x$, so that $y_1/y_2 = 1/k = \text{const}$. Then we can write $y = c_1 y_1 + c_2 y_2$ in the form

$$y = c_1 \cos x + c_2(k \cos x) = C \cos x \qquad \text{where} \qquad C = c_1 + c_2 k.$$

Hence we are no longer able to satisfy two initial conditions with only one arbitrary constant C. Consequently, in defining the concept of a general solution, we must exclude proportionality. And we see at the same time why the concept of a general solution is of importance in connection with initial value problems.

DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of an ODE (2) on an open interval I is a solution (5) in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants. These y_1, y_2 are called a **basis** (or a **fundamental system**) of solutions of (2) on I.

A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (5).

For the definition of an *interval* see Sec. 1.1. Furthermore, as usual, y_1 and y_2 are called *proportional* on I if for all x on I,

$$(6) \qquad (a) \quad y_1 = ky_2 \qquad \text{or} \qquad (b) \quad y_2 = ly_1$$

where k and l are numbers, zero or not. (Note that (a) implies (b) if and only if $k \neq 0$).

Actually, we can reformulate our definition of a basis by using a concept of general importance. Namely, two functions y_1 and y_2 are called **linearly independent** on an interval I where they are defined if

$$(7) \qquad k_1y_1(x) + k_2y_2(x) = 0 \qquad \text{everywhere on } I \text{ implies} \qquad k_1 = 0 \text{ and } k_2 = 0.$$

And y_1 and y_2 are called **linearly dependent** on I if (7) also holds for some constants k_1, k_2 not both zero. Then, if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1}y_2 \qquad \text{or} \qquad y_2 = -\frac{k_1}{k_2}y_1.$$

In contrast, in the case of linear *independence* these functions are not proportional because then we cannot divide in (7). This gives the following

DEFINITION

Basis (Reformulated)

A **basis** of solutions of (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

If the coefficients p and q of (2) are continuous on some open interval I, then (2) has a general solution. It yields the unique solution of any initial value problem (2), (4). It includes all solutions of (2) on I; hence (2) has no *singular solutions* (solutions not obtainable from of a general solution; see also Problem Set 1.1). All this will be shown in Sec. 2.6.

EXAMPLE 5 **Basis, General Solution, Particular Solution**

$\cos x$ and $\sin x$ in Example 4 form a basis of solutions of the ODE $y'' + y = 0$ for all x because their quotient is $\cot x \neq \text{const}$ (or $\tan x \neq \text{const}$). Hence $y = c_1 \cos x + c_2 \sin x$ is a general solution. The solution $y = 3.0 \cos x - 0.5 \sin x$ of the initial value problem is a particular solution. ■

EXAMPLE 6 **Basis, General Solution, Particular Solution**

Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y'' - y = 0, \qquad y(0) = 6, \qquad y'(0) = -2.$$

Solution. $(e^x)'' - e^x = 0$ and $(e^{-x})'' - e^{-x} = 0$ show that e^x and e^{-x} are solutions. They are not proportional, $e^x/e^{-x} = e^{2x} \neq \text{const}$. Hence e^x, e^{-x} form a basis for all x. We now write down the corresponding general solution and its derivative and equate their values at 0 to the given initial conditions,

$$y = c_1 e^x + c_2 e^{-x}, \qquad y' = c_1 e^x - c_2 e^{-x}, \qquad y(0) = c_1 + c_2 = 6, \qquad y'(0) = c_1 - c_2 = -2.$$

By addition and subtraction, $c_1 = 2$, $c_2 = 4$, so that the *answer* is $y = 2e^x + 4e^{-x}$. This is the particular solution satisfying the two initial conditions. ■

Find a Basis if One Solution Is Known. Reduction of Order

It happens quite often that one solution can be found by inspection or in some other way. Then a second linearly independent solution can be obtained by solving a first-order ODE. This is called the method of **reduction of order**.[1] We first show how this method works in an example and then in general.

EXAMPLE 7 **Reduction of Order if a Solution Is Known. Basis**

Find a basis of solutions of the ODE

$$(x^2 - x)y'' - xy' + y = 0.$$

Solution. Inspection shows that $y_1 = x$ is a solution because $y_1' = 1$ and $y_1'' = 0$, so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

$$y = uy_1 = ux, \qquad y' = u'x + u, \qquad y'' = u''x + 2u'$$

into the ODE. This gives

$$(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.$$

ux and $-xu$ cancel and we are left with the following ODE, which we divide by x, order, and simplify,

$$(x^2 - x)(u''x + 2u') - x^2u' = 0, \qquad (x^2 - x)u'' + (x - 2)u' = 0.$$

This ODE is of first order in $v = u'$, namely, $(x^2 - x)v' + (x - 2)v = 0$. Separation of variables and integration gives

$$\frac{dv}{v} = -\frac{x - 2}{x^2 - x}\,dx = \left(\frac{1}{x - 1} - \frac{2}{x}\right)dx, \qquad \ln|v| = \ln|x - 1| - 2\ln|x| = \ln\frac{|x - 1|}{x^2}.$$

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[1] Credited to the great mathematician JOSEPH LOUIS LAGRANGE (1736–1813), who was born in Turin, of French extraction, got his first professorship when he was 19 (at the Military Academy of Turin), became director of the mathematical section of the Berlin Academy in 1766, and moved to Paris in 1787. His important major work was in the calculus of variations, celestial mechanics, general mechanics (*Mécanique analytique,* Paris, 1788), differential equations, approximation theory, algebra, and number theory.

We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

$$v = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \qquad u = \int v\,dx = \ln|x| + \frac{1}{x}, \qquad \text{hence} \qquad y_2 = ux = x\ln|x| + 1.$$

Since $y_1 = x$ and $y_2 = x\ln|x| + 1$ are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive x. ■

In this example we applied **reduction of order** to a homogeneous linear ODE [see (2)]

$$y'' + p(x)y' + q(x)y = 0.$$

Note that we now take the ODE in standard form, with y'', not $f(x)y''$—this is essential in applying our subsequent formulas. We assume a solution y_1 of (2), on an open interval I, to be known and want to find a basis. For this we need a second linearly independent solution y_2 of (2) on I. To get y_2, we substitute

$$y = y_2 = uy_1, \qquad y' = y_2' = u'y_1 + uy_1', \qquad y'' = y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

into (2). This gives

$$u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0. \tag{8}$$

Collecting terms in u'', u', and u, we have

$$u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0.$$

Now comes the main point. Since y_1 is a solution of (2), the expression in the last parentheses is zero. Hence u is gone, and we are left with an ODE in u' and u''. We divide this remaining ODE by y_1 and set $u' = U, u'' = U'$,

$$u'' + u'\frac{2y_1' + py_1}{y_1} = 0, \qquad \text{thus} \qquad U' + \left(\frac{2y_1'}{y_1} + p\right)U = 0.$$

This is the desired first-order ODE, the reduced ODE. Separation of variables and integration gives

$$\frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right)dx \qquad \text{and} \qquad \ln|U| = -2\ln|y_1| - \int p\,dx.$$

By taking exponents we finally obtain

$$\boxed{U = \frac{1}{y_1^2}e^{-\int p\,dx}.} \tag{9}$$

Here $U = u'$, so that $u = \int U\,dx$. Hence the desired second solution is

$$y_2 = y_1u = y_1\int U\,dx.$$

The quotient $y_2/y_1 = u = \int U\,dx$ cannot be constant (since $U > 0$), so that y_1 and y_2 form a basis of solutions.

PROBLEM SET 2.1

REDUCTION OF ORDER is important because it gives a simpler ODE. A general second-order ODE $F(x, y, y', y'') = 0$, linear or not, can be reduced to first order if y does not occur explicitly (Prob. 1) or if x does not occur explicitly (Prob. 2) or if the ODE is homogeneous linear and we know a solution (see the text).

1. **Reduction.** Show that $F(x, y', y'') = 0$ can be reduced to first order in $z = y'$ (from which y follows by integration). Give two examples of your own.
2. **Reduction.** Show that $F(y, y', y'') = 0$ can be reduced to a first-order ODE with y as the independent variable and $y'' = (dz/dy)z$, where $z = y'$; derive this by the chain rule. Give two examples.

3–10 REDUCTION OF ORDER

Reduce to first order and solve, showing each step in detail.

3. $y'' + y' = 0$
4. $2xy'' = 3y'$
5. $yy'' = 3y'^2$
6. $xy'' + 2y' + xy = 0, \quad y_1 = (\cos x)/x$
7. $y'' + y'^3 \sin y = 0$
8. $y'' = 1 + y'^2$
9. $x^2y'' - 5xy' + 9y = 0, \quad y_1 = x^3$
10. $y'' + (1 + 1/y)y'^2 = 0$

11–14 APPLICATIONS OF REDUCIBLE ODEs

11. **Curve.** Find the curve through the origin in the xy-plane which satisfies $y'' = 2y'$ and whose tangent at the origin has slope 1.
12. **Hanging cable.** It can be shown that the curve $y(x)$ of an inextensible flexible homogeneous cable hanging between two fixed points is obtained by solving $y'' = k\sqrt{1 + y'^2}$, where the constant k depends on the weight. This curve is called *catenary* (from Latin *catena* = the chain). Find and graph $y(x)$, assuming that $k = 1$ and those fixed points are $(-1, 0)$ and $(1, 0)$ in a vertical xy-plane.
13. **Motion.** If, in the motion of a small body on a straight line, the sum of velocity and acceleration equals a positive constant, how will the distance $y(t)$ depend on the initial velocity and position?
14. **Motion.** In a straight-line motion, let the velocity be the reciprocal of the acceleration. Find the distance $y(t)$ for arbitrary initial position and velocity.

15–19 GENERAL SOLUTION. INITIAL VALUE PROBLEM (IVP)

(More in the next set.) **(a)** Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. **(b)** Solve the IVP. Graph or sketch the solution.

15. $4y'' + 25y = 0, \quad y(0) = 3.0, \quad y'(0) = -2.5,$ $\cos 2.5x, \sin 2.5x$
16. $y'' + 0.6y' + 0.09y = 0, \quad y(0) = 2.2, \quad y'(0) = 0.14,$ $e^{-0.3x}, xe^{-0.3x}$
17. $4x^2y'' - 3y = 0, \quad y(1) = -3, \quad y'(1) = 0,$ $x^{3/2}, x^{-1/2}$
18. $x^2y'' - xy' + y = 0, \quad y(1) = 4.3, \quad y'(1) = 0.5,$ $x, x \ln x$
19. $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 15,$ $e^{-x} \cos x, e^{-x} \sin x$
20. **CAS PROJECT. Linear Independence.** Write a program for testing linear independence and dependence. Try it out on some of the problems in this and the next problem set and on examples of your own.

2.2 Homogeneous Linear ODEs with Constant Coefficients

We shall now consider second-order homogeneous linear ODEs whose coefficients a and b are constant,

$$y'' + ay' + by = 0. \tag{1}$$

These equations have important applications in mechanical and electrical vibrations, as we shall see in Secs. 2.4, 2.8, and 2.9.

To solve (1), we recall from Sec. 1.5 that the solution of the first-order linear ODE with a constant coefficient k

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

$$y = e^{\lambda x}. \tag{2}$$

Substituting (2) and its derivatives

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

into our equation (1), we obtain

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.$$

Hence if λ is a solution of the important **characteristic equation** (or *auxiliary equation*)

$$\lambda^2 + a\lambda + b = 0 \tag{3}$$

then the exponential function (2) is a solution of the ODE (1). Now from algebra we recall that the roots of this quadratic equation (3) are

$$\lambda_1 = \tfrac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right), \qquad \lambda_2 = \tfrac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right). \tag{4}$$

(3) and (4) will be basic because our derivation shows that the functions

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x} \tag{5}$$

are solutions of (1). Verify this by substituting (5) into (1).

From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I) *Two real roots if* $a^2 - 4b > 0$,
(Case II) *A real double root if* $a^2 - 4b = 0$,
(Case III) *Complex conjugate roots if* $a^2 - 4b < 0$.

Case I. Two Distinct Real-Roots λ_1 and λ_2

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

because y_1 and y_2 are defined (and real) for all x and their quotient is not constant. The corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \tag{6}$$

EXAMPLE 1 **General Solution in the Case of Distinct Real Roots**

We can now solve $y'' - y = 0$ in Example 6 of Sec. 2.1 systematically. The characteristic equation is $\lambda^2 - 1 = 0$. Its roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence a basis of solutions is e^x and e^{-x} and gives the same general solution as before,

$$y = c_1 e^x + c_2 e^{-x}.$$

EXAMPLE 2 **Initial Value Problem in the Case of Distinct Real Roots**

Solve the initial value problem

$$y'' + y' - 2y = 0, \qquad y(0) = 4, \qquad y'(0) = -5.$$

Solution. ***Step 1. General solution.*** The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0.$$

Its roots are

$$\lambda_1 = \tfrac{1}{2}(-1 + \sqrt{9}) = 1 \qquad \text{and} \qquad \lambda_2 = \tfrac{1}{2}(-1 - \sqrt{9}) = -2$$

so that we obtain the general solution

$$y = c_1 e^x + c_2 e^{-2x}.$$

Step 2. Particular solution. Since $y'(x) = c_1 e^x - 2c_2 e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$
$$y'(0) = c_1 - 2c_2 = -5.$$

Hence $c_1 = 1$ and $c_2 = 3$. This gives the *answer* $y = e^x + 3e^{-2x}$. Figure 30 shows that the curve begins at $y = 4$ with a negative slope (-5, but note that the axes have different scales!), in agreement with the initial conditions.

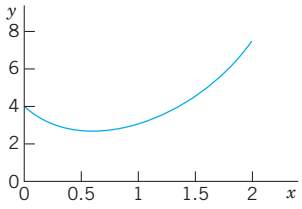


Fig. 30. Solution in Example 2

Case II. Real Double Root $\lambda = -a/2$

If the discriminant $a^2 - 4b$ is zero, we see directly from (4) that we get only one root, $\lambda = \lambda_1 = \lambda_2 = -a/2$, hence only one solution,

$$y_1 = e^{-(a/2)x}.$$

To obtain a second independent solution y_2 (needed for a basis), we use the method of reduction of order discussed in the last section, setting $y_2 = uy_1$. Substituting this and its derivatives $y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + 2u'y_1' + uy_1''$ into (1), we first have

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0.$$

Collecting terms in u'', u', and u, as in the last section, we obtain

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0.$$

The expression in the last parentheses is zero, since y_1 is a solution of (1). The expression in the first parentheses is zero, too, since

$$2y_1' = -ae^{-ax/2} = -ay_1.$$

We are thus left with $u''y_1 = 0$. Hence $u'' = 0$. By two integrations, $u = c_1x + c_2$. To get a second independent solution $y_2 = uy_1$, we can simply choose $c_1 = 1$, $c_2 = 0$ and take $u = x$. Then $y_2 = xy_1$. Since these solutions are not proportional, they form a basis. Hence in the case of a double root of (3) a basis of solutions of (1) on any interval is

$$e^{-ax/2}, \qquad xe^{-ax/2}.$$

The corresponding general solution is

(7) $$y = (c_1 + c_2x)e^{-ax/2}.$$

WARNING! If λ is a *simple* root of (4), then $(c_1 + c_2x)e^{\lambda x}$ with $c_2 \neq 0$ is ***not*** a solution of (1).

EXAMPLE 3 General Solution in the Case of a Double Root

The characteristic equation of the ODE $y'' + 6y' + 9y = 0$ is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. It has the double root $\lambda = -3$. Hence a basis is e^{-3x} and xe^{-3x}. The corresponding general solution is $y = (c_1 + c_2x)e^{-3x}$. ■

EXAMPLE 4 Initial Value Problem in the Case of a Double Root

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Solution. The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2x)e^{-0.5x}.$$

We need its derivative

$$y' = c_2e^{-0.5x} - 0.5(c_1 + c_2x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0, \qquad y'(0) = c_2 - 0.5c_1 = 3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$. See Fig. 31. ■

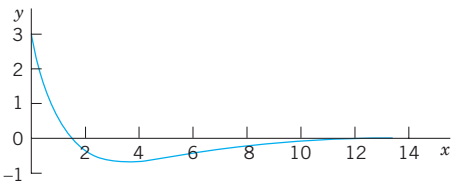


Fig. 31. Solution in Example 4

Case III. Complex Roots $-\frac{1}{2}a + i\omega$ and $-\frac{1}{2}a - i\omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) are the complex $\lambda = -\frac{1}{2}a \pm i\omega$ that give the complex solutions of the ODE (1). However, we will show that we can obtain a basis of *real* solutions

$$(8) \qquad y_1 = e^{-ax/2}\cos \omega x, \qquad y_2 = e^{-ax/2}\sin \omega x \qquad (\omega > 0)$$

where $\omega^2 = b - \frac{1}{4}a^2$. It can be verified by substitution that these are solutions in the present case. We shall derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient $\cot \omega x$ is not constant. Hence a real general solution in Case III is

$$\mathbf{(9)} \qquad y = e^{-ax/2}(A\cos \omega x + B\sin \omega x) \qquad (A, B \text{ arbitrary}).$$

EXAMPLE 5 **Complex Roots. Initial Value Problem**

Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution. ***Step 1. General solution.*** The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A\cos 3x + B\sin 3x).$$

Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x}\sin 3x$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x).$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

$$y = e^{-0.2x}\sin 3x.$$

Figure 32 shows y and the curves of $e^{-0.2x}$ and $-e^{-0.2x}$ (dashed), between which the curve of y oscillates. Such "damped vibrations" (with $x = t$ being time) have important mechanical and electrical applications, as we shall soon see (in Sec. 2.4). ■

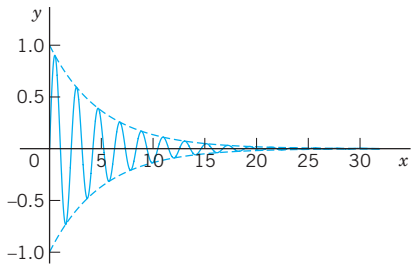


Fig. 32. Solution in Example 5

EXAMPLE 6 **Complex Roots**

A general solution of the ODE

$$y'' + \omega^2 y = 0 \qquad (\omega \text{ constant, not zero})$$

is

$$y = A\cos \omega x + B\sin \omega x.$$

With $\omega = 1$ this confirms Example 4 in Sec. 2.1. ■

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos \omega x$ $e^{-ax/2}\sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

It is very interesting that in applications to mechanical systems or electrical circuits, these three cases correspond to three different forms of motion or flows of current, respectively. We shall discuss this basic relation between theory and practice in detail in Sec. 2.4 (and again in Sec. 2.8).

Derivation in Case III. Complex Exponential Function

If verification of the solutions in (8) satisfies you, skip the systematic derivation of these real solutions from the complex solutions by means of the complex exponential function e^z of a complex variable $z = r + it$. We write $r + it$, not $x + iy$ because x and y occur in the ODE. The definition of e^z in terms of the real functions e^r, $\cos t$, and $\sin t$ is

$$e^z = e^{r+it} = e^r e^{it} = e^r(\cos t + i \sin t). \tag{10}$$

This is motivated as follows. For real $z = r$, hence $t = 0$, $\cos 0 = 1$, $\sin 0 = 0$, we get the real exponential function e^r. It can be shown that $e^{z_1+z_2} = e^{z_1}e^{z_2}$, just as in real. (Proof in Sec. 13.5.) Finally, if we use the Maclaurin series of e^z with $z = it$ as well as $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc., and reorder the terms as shown (this is permissible, as can be proved), we obtain the series

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \cdots \\ &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - + \cdots + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \cdots\right) \\ &= \cos t + i \sin t. \end{aligned}$$

(Look up these real series in your calculus book if necessary.) We see that we have obtained the formula

$$\boxed{e^{it} = \cos t + i \sin t,} \tag{11}$$

called the **Euler formula**. Multiplication by e^r gives (10).

For later use we note that $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$, so that by addition and subtraction of this and (11),

$$(12) \qquad \cos t = \tfrac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}).$$

After these comments on the definition (10), let us now turn to Case III.

In Case III the radicand $a^2 - 4b$ in (4) is negative. Hence $4b - a^2$ is positive and, using $\sqrt{-1} = i$, we obtain in (4)

$$\tfrac{1}{2}\sqrt{a^2 - 4b} = \tfrac{1}{2}\sqrt{-(4b - a^2)} = \sqrt{-(b - \tfrac{1}{4}a^2)} = i\sqrt{b - \tfrac{1}{4}a^2} = i\omega$$

with ω defined as in (8). Hence in (4),

$$\lambda_1 = \tfrac{1}{2}a + i\omega \qquad \text{and, similarly,} \qquad \lambda_2 = \tfrac{1}{2}a - i\omega.$$

Using (10) with $r = -\frac{1}{2}ax$ and $t = \omega x$, we thus obtain

$$e^{\lambda_1 x} = e^{-(a/2)x + i\omega x} = e^{-(a/2)x}(\cos \omega x + i \sin \omega x)$$
$$e^{\lambda_2 x} = e^{-(a/2)x - i\omega x} = e^{-(a/2)x}(\cos \omega x - i \sin \omega x).$$

We now add these two lines and multiply the result by $\frac{1}{2}$. This gives y_1 as in (8). Then we subtract the second line from the first and multiply the result by $1/(2i)$. This gives y_2 as in (8). These results obtained by addition and multiplication by constants are again solutions, as follows from the superposition principle in Sec. 2.1. This concludes the derivation of these real solutions in Case III.

PROBLEM SET 2.2

1–15 GENERAL SOLUTION

Find a general solution. Check your answer by substitution. ODEs of this kind have important applications to be discussed in Secs. 2.4, 2.7, and 2.9.

1. $4y'' - 25y = 0$
2. $y'' + 36y = 0$
3. $y'' + 6y' + 8.96y = 0$
4. $y'' + 4y' + (\pi^2 + 4)y = 0$
5. $y'' + 2\pi y' + \pi^2 y = 0$
6. $10y'' - 32y' + 25.6y = 0$
7. $y'' + 4.5y' = 0$
8. $y'' + y' + 3.25y = 0$
9. $y'' + 1.8y' - 2.08y = 0$
10. $100y'' + 240y' + (196\pi^2 + 144)y = 0$
11. $4y'' - 4y' - 3y = 0$
12. $y'' + 9y' + 20y = 0$
13. $9y'' - 30y' + 25y = 0$
14. $y'' + 2k^2y' + k^4y = 0$
15. $y'' + 0.54y' + (0.0729 + \pi)y = 0$

16–20 FIND AN ODE

$y'' + ay' + by = 0$ for the given basis.

16. $e^{2.6x}$, $e^{-4.3x}$
17. $e^{-\sqrt{5}x}$, $xe^{-\sqrt{5}x}$
18. $\cos 2\pi x$, $\sin 2\pi x$
19. $e^{(-2+i)x}$, $e^{(-2-i)x}$
20. $e^{-3.1x}\cos 2.1x$, $e^{-3.1x}\sin 2.1x$

21–30 INITIAL VALUES PROBLEMS

Solve the IVP. Check that your answer satisfies the ODE as well as the initial conditions. Show the details of your work.

21. $y'' + 25y = 0$, $y(0) = 4.6$, $y'(0) = -1.2$
22. The ODE in Prob. 4, $y(\frac{1}{2}) = 1$, $y'(\frac{1}{2}) = -2$
23. $y'' + y' - 6y = 0$, $y(0) = 10$, $y'(0) = 0$
24. $4y'' - 4y' - 3y = 0$, $y(-2) = e$, $y'(-2) = -e/2$
25. $y'' - y = 0$, $y(0) = 2$, $y'(0) = -2$
26. $y'' - k^2y = 0$ $(k \neq 0)$, $y(0) = 1$, $y'(0) = 1$

27. The ODE in Prob. 5,
$y(0) = 4.5, \quad y'(0) = -4.5\pi - 1 = 13.137$

28. $8y'' - 2y' - y = 0, \quad y(0) = -0.2, \quad y'(0) = -0.325$

29. The ODE in Prob. 15, $y(0) = 0, \quad y'(0) = 1$

30. $9y'' - 30y' + 25y = 0, \quad y(0) = 3.3, \quad y'(0) = 10.0$

31–36 LINEAR INDEPENDENCE is of basic importance, in this chapter, in connection with general solutions, as explained in the text. Are the following functions linearly independent on the given interval? Show the details of your work.

31. e^{kx}, xe^{kx}, any interval

32. $e^{ax}, e^{-ax}, \quad x > 0$

33. $x^2, x^2 \ln x, \quad x > 1$

34. $\ln x, \ln (x^3), \quad x > 1$

35. $\sin 2x, \cos x \sin x, \quad x < 0$

36. $e^{-x} \cos \frac{1}{2}x, 0, \quad -1 \leqq x \leqq 1$

37. Instability. Solve $y'' - y = 0$ for the initial conditions $y(0) = 1, y'(0) = -1$. Then change the initial conditions to $y(0) = 1.001, y'(0) = -0.999$ and explain why this small change of 0.001 at $t = 0$ causes a large change later, e.g., 22 at $t = 10$. This is instability: a small initial difference in setting a quantity (a current, for instance) becomes larger and larger with time t. This is undesirable.

38. TEAM PROJECT. General Properties of Solutions

(a) Coefficient formulas. Show how a and b in (1) can be expressed in terms of λ_1 and λ_2. Explain how these formulas can be used in constructing equations for given bases.

(b) Root zero. Solve $y'' + 4y' = 0$ (i) by the present method, and (ii) by reduction to first order. Can you explain why the result must be the same in both cases? Can you do the same for a general ODE $y'' + ay' = 0$?

(c) Double root. Verify directly that $xe^{\lambda x}$ with $\lambda = -a/2$ is a solution of (1) in the case of a double root. Verify and explain why $y = e^{-2x}$ is a solution of $y'' - y' - 6y = 0$ but xe^{-2x} is not.

(d) Limits. Double roots should be limiting cases of distinct roots λ_1, λ_2 as, say, $\lambda_2 \to \lambda_1$. Experiment with this idea. (Remember l'Hôpital's rule from calculus.) Can you arrive at $xe^{\lambda_1 x}$? Give it a try.

2.3 Differential Operators. *Optional*

This short section can be omitted without interrupting the flow of ideas. It will not be used subsequently, except for the notations Dy, D^2y, etc. to stand for y', y'', etc.

Operational calculus means the technique and application of operators. Here, an **operator** is a transformation that transforms a function into another function. Hence differential calculus involves an operator, the **differential operator** D, which transforms a (differentiable) function into its derivative. In operator notation we write $D = \frac{d}{dx}$ and

$$(1) \qquad Dy = y' = \frac{dy}{dx}.$$

Similarly, for the higher derivatives we write $D^2y = D(Dy) = y''$, and so on. For example, $D \sin = \cos, D^2 \sin = -\sin$, etc.

For a homogeneous linear ODE $y'' + ay' + by = 0$ with constant coefficients we can now introduce the **second-order differential operator**

$$L = P(D) = D^2 + aD + bI,$$

where I is the **identity operator** defined by $Iy = y$. Then we can write that ODE as

$$(2) \qquad Ly = P(D)y = (D^2 + aD + bI)y = 0.$$

P suggests "polynomial." L is a **linear operator**. By definition this means that if Ly and Lw exist (this is the case if y and w are twice differentiable), then $L(cy + kw)$ exists for any constants c and k, and

$$L(cy + kw) = cLy + kLw.$$

Let us show that from (2) we reach agreement with the results in Sec. 2.2. Since $(De^{\lambda})(x) = \lambda e^{\lambda x}$ and $(D^2e^{\lambda})(x) = \lambda^2 e^{\lambda x}$, we obtain

$$\begin{aligned} Le^{\lambda}(x) &= P(D)e^{\lambda}(x) = (D^2 + aD + bI)e^{\lambda}(x) \\ &= (\lambda^2 + a\lambda + b)e^{\lambda x} = P(\lambda)e^{\lambda x} = 0. \end{aligned} \tag{3}$$

This confirms our result of Sec. 2.2 that $e^{\lambda x}$ *is a solution of the ODE* (2) *if and only if* λ *is a solution of the characteristic equation* $P(\lambda) = 0$.

$P(\lambda)$ is a polynomial in the usual sense of algebra. If we replace λ by the operator D, we obtain the "operator polynomial" $P(D)$. *The point of this operational calculus is that* $P(D)$ *can be treated just like an algebraic quantity.* In particular, we can factor it.

EXAMPLE 1 **Factorization, Solution of an ODE**

Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$.

Solution. $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $I^2 = I$. Now $(D - 8I)y = y' - 8y = 0$ has the solution $y_1 = e^{8x}$. Similarly, the solution of $(D + 5I)y = 0$ is $y_2 = e^{-5x}$. This is a basis of $P(D)y = 0$ on any interval. From the factorization we obtain the ODE, as expected,

$$\begin{aligned} (D - 8I)(D + 5I)y = (D - 8I)(y' + 5y) &= D(y' + 5y) - 8(y' + 5y) \\ &= y'' + 5y' - 8y' - 40y = y'' - 3' - 40y = 0. \end{aligned}$$

Verify that this agrees with the result of our method in Sec. 2.2. This is not unexpected because we factored $P(D)$ in the same way as the characteristic polynomial $P(\lambda) = \lambda^2 - 3\lambda - 40$. ■

It was essential that L in (2) had *constant* coefficients. Extension of operator methods to variable-coefficient ODEs is more difficult and will not be considered here.

If operational methods were limited to the simple situations illustrated in this section, it would perhaps not be worth mentioning. Actually, the power of the operator approach appears in more complicated engineering problems, as we shall see in Chap. 6.

PROBLEM SET 2.3

1–5 APPLICATION OF DIFFERENTIAL OPERATORS

Apply the given operator to the given functions. Show all steps in detail.

1. $D^2 + 2D$; $\cosh 2x, e^{-x} + e^{2x}, \cos x$
2. $D - 3I$; $3x^2 + 3x, 3e^{3x}, \cos 4x - \sin 4x$
3. $(D - 2I)^2$; e^{2x}, xe^{2x}, e^{-2x}
4. $(D + 6I)^2$; $6x + \sin 6x, xe^{-6x}$
5. $(D - 2I)(D + 3I)$; e^{2x}, xe^{2x}, e^{-3x}

6–12 GENERAL SOLUTION

Factor as in the text and solve.

6. $(D^2 + 4.00D + 3.36I)y = 0$
7. $(4D^2 - I)y = 0$
8. $(D^2 + 3I)y = 0$
9. $(D^2 - 4.20D + 4.41I)y = 0$
10. $(D^2 + 4.80D + 5.76I)y = 0$
11. $(D^2 - 4.00D + 3.84I)y = 0$
12. $(D^2 + 3.0D + 2.5I)y = 0$

13. **Linear operator.** Illustrate the linearity of L in (2) by taking $c = 4, k = -6, y = e^{2x}$, and $w = \cos 2x$. Prove that L is linear.

14. **Double root.** If $D^2 + aD + bI$ has distinct roots μ and λ, show that a particular solution is $y = (e^{\mu x} - e^{\lambda x})/(\mu - \lambda)$. Obtain from this a solution $xe^{\lambda x}$ by letting $\mu \to \lambda$ and applying l'Hôpital's rule.

15. **Definition of linearity.** Show that the definition of linearity in the text is equivalent to the following. If $L[y]$ and $L[w]$ exist, then $L[y + w]$ exists and $L[cy]$ and $L[kw]$ exist for all constants c and k, and $L[y + w] = L[y] + L[w]$ as well as $L[cy] = cL[y]$ and $L[kw] = kL[w]$.

2.4 Modeling of Free Oscillations of a Mass–Spring System

Linear ODEs with constant coefficients have important applications in mechanics, as we show in this section as well as in Sec. 2.8, and in electrical circuits as we show in Sec. 2.9. In this section we model and solve a basic mechanical system consisting of a mass on an elastic spring (a so-called "mass–spring system," Fig. 33), which moves up and down.

Setting Up the Model

We take an ordinary coil spring that resists extension as well as compression. We suspend it vertically from a fixed support and attach a body at its lower end, for instance, an iron ball, as shown in Fig. 33. We let $y = 0$ denote the position of the ball when the system is at rest (Fig. 33b). Furthermore, we choose ***the downward direction as positive***, thus regarding downward forces as *positive* and upward forces as *negative*.

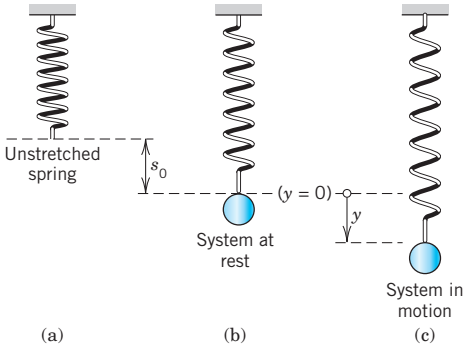


Fig. 33. Mechanical mass–spring system

We now let the ball move, as follows. We pull it down by an amount $y > 0$ (Fig. 33c). This causes a spring force

$$F_1 = -ky \tag{1}$$

(Hooke's law[2])

proportional to the stretch y, with $k\,(>0)$ called the **spring constant**. The minus sign indicates that F_1 points upward, against the displacement. It is a *restoring force*: It wants to restore the system, that is, to pull it back to $y = 0$. Stiff springs have large k.

[2]ROBERT HOOKE (1635–1703), English physicist, a forerunner of Newton with respect to the law of gravitation.

Note that an additional force $-F_0$ is present in the spring, caused by stretching it in fastening the ball, but F_0 has no effect on the motion because it is in equilibrium with the weight W of the ball, $-F_0 = W = mg$, where $g = 980\ \text{cm/sec}^2 = 9.8\ \text{m/sec}^2 = 32.17\ \text{ft/sec}^2$ is the **constant of gravity at the Earth's surface** (not to be confused with the *universal gravitational constant* $G = gR^2/M = 6.67 \cdot 10^{-11}\ \text{nt m}^2/\text{kg}^2$, which we shall not need; here $R = 6.37 \cdot 10^6\ \text{m}$ and $M = 5.98 \cdot 10^{24}\ \text{kg}$ are the Earth's radius and mass, respectively).

The motion of our mass–spring system is determined by **Newton's second law**

$$\textbf{(2)} \qquad \text{Mass} \times \text{Acceleration} = my'' = \text{Force}$$

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the ball. (For systems of units, see the inside of the front cover.)

ODE of the Undamped System

Every system has damping. Otherwise it would keep moving forever. But if the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping. Then Newton's law with $F = -F_1$ gives the model $my'' = -F_1 = -ky$; thus

$$\textbf{(3)} \qquad my'' + ky = 0.$$

This is a homogeneous linear ODE with constant coefficients. A general solution is obtained as in Sec. 2.2, namely (see Example 6 in Sec. 2.2)

$$\textbf{(4)} \qquad y(t) = A \cos \omega_0 t + B \sin \omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

This motion is called a **harmonic oscillation** (Fig. 34). Its *frequency* is $f = \omega_0/2\pi$ Hertz[3] (= cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$. The frequency f is called the **natural frequency** of the system. (We write ω_0 to reserve ω for Sec. 2.8.)

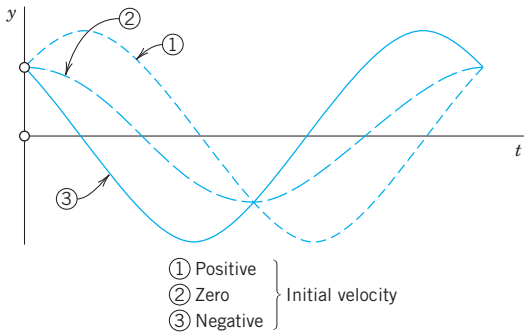


Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

[3]**HEINRICH HERTZ (1857–1894)**, German physicist, who discovered electromagnetic waves, as the basis of wireless communication developed by GUGLIELMO MARCONI (1874–1937), Italian physicist (Nobel prize in 1909).

An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

(4*) $$y(t) = C\cos(\omega_0 t - \delta)$$

with $C = \sqrt{A^2 + B^2}$ and phase angle δ, where $\tan\delta = B/A$. This follows from the addition formula (6) in App. 3.1.

EXAMPLE 1 **Harmonic Oscillation of an Undamped Mass–Spring System**

If a mass–spring system with an iron ball of weight $W = 98$ nt (about 22 lb) can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m (about 43 in.), how many cycles per minute will the system execute? What will its motion be if we pull the ball down from rest by 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke's law (1) with W as the force and 1.09 meter as the stretch gives $W = 1.09k$; thus $k = W/1.09 = 98/1.09 = 90$ [kg/sec^2] $= 90$ [nt/meter]. The mass is $m = W/g = 98/9.8 = 10$ [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] $= 29$ [cycles/min].

From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16\cos 3t \text{ [meter]} \quad \text{or} \quad 0.52\cos 3t \text{ [ft]} \qquad \text{(Fig. 35).}$$

If you have a chance of experimenting with a mass–spring system, don't miss it. You will be surprised about the good agreement between theory and experiment, usually within a fraction of one percent if you measure carefully. ■

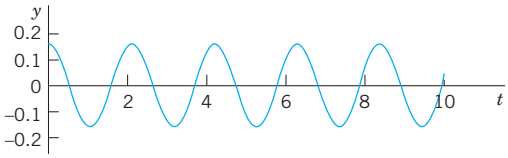


Fig. 35. Harmonic oscillation in Example 1

ODE of the Damped System

To our model $my'' = -ky$ we now add a damping force

$$F_2 = -cy',$$

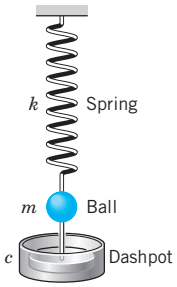


Fig. 36. Damped system

obtaining $my'' = -ky - cy'$; thus the ODE of the damped mass–spring system is

(5) $$my'' + cy' + ky = 0. \qquad \text{(Fig. 36)}$$

Physically this can be done by connecting the ball to a dashpot; see Fig. 36. We assume this damping force to be proportional to the velocity $y' = dy/dt$. This is generally a good approximation for small velocities.

The constant c is called the *damping constant.* Let us show that c is positive. Indeed, the damping force $F_2 = -cy'$ acts *against* the motion; hence for a downward motion we have $y' > 0$ which for positive c makes F negative (an upward force), as it should be. Similarly, for an upward motion we have $y' < 0$ which, for $c > 0$ makes F_2 positive (a downward force).

The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by m)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$(6)\quad \lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case I.	$c^2 > 4mk$.	*Distinct real roots* λ_1, λ_2.	**(Overdamping)**
Case II.	$c^2 = 4mk$.	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk$.	*Complex conjugate roots.*	**(Underdamping)**

They correspond to the three Cases I, II, III in Sec. 2.2.

Discussion of the Three Cases

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots. In this case the corresponding general solution of (5) is

$$(7)\quad y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0, \beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the *static equilibrium position* ($y = 0$). Figure 37 shows (7) for some typical initial conditions.

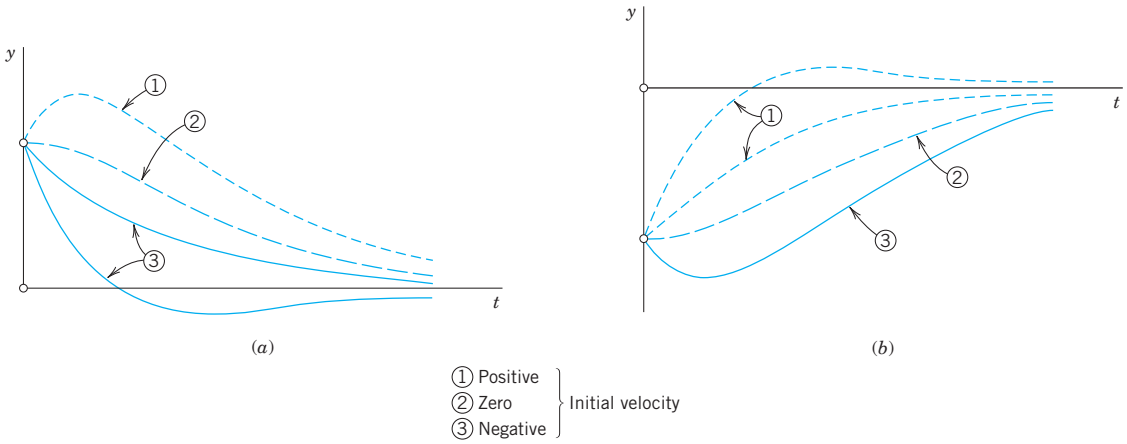


Fig. 37. Typical motions (7) in the overdamped case
(*a*) Positive initial displacement
(*b*) Negative initial displacement

Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

(8) $$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$

This solution can pass through the equilibrium position $y = 0$ at most once because $e^{-\alpha t}$ is never zero and $c_1 + c_2 t$ can have at most one positive zero. If both c_1 and c_2 are positive (or both negative), it has no positive zero, so that y does not pass through 0 at all. Figure 38 shows typical forms of (8). Note that they look almost like those in the previous figure.

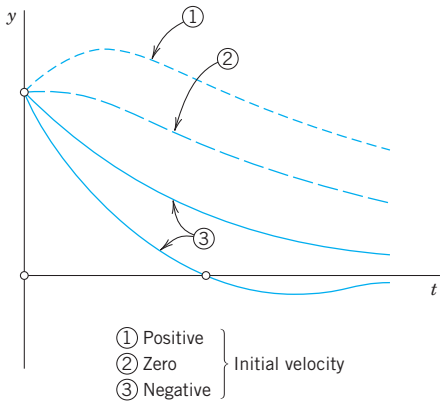


Fig. 38. Critical damping [see (8)]

Case III. Underdamping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$(9) \qquad \beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0).$$

(We now write ω^* to reserve ω for driving and electromotive forces in Secs. 2.8 and 2.9.) The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$\textbf{(10)} \qquad y(t) = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t) = Ce^{-\alpha t}\cos(\omega^* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan\delta = B/A$, as in (4*).

This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which $\cos(\omega^* t - \delta)$ equals 1 or -1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (>0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

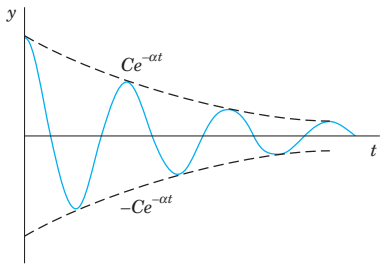


Fig. 39. Damped oscillation in Case III [see (10)]

EXAMPLE 2 **The Three Cases of Damped Motion**

How does the motion in Example 1 change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

$$\text{(I) } c = 100 \text{ kg/sec}, \qquad \text{(II) } c = 60 \text{ kg/sec}, \qquad \text{(III) } c = 10 \text{ kg/sec}.$$

Solution. It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

(I) With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$. It has the roots -9 and -1. This gives the general solution

$$y = c_1e^{-9t} + c_2e^{-t}. \qquad \text{We also need} \qquad y' = -9c_1e^{-9t} - c_2e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

It approaches 0 as $t \to \infty$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root -3. Hence the corresponding general solution is

$$y = (c_1 + c_2t)e^{-3t}. \qquad \text{We also need} \qquad y' = (c_2 - 3c_1 - 3c_2t)e^{-3t}.$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

(III) The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical c, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

This gives the general solution

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0$, $B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 40. ■

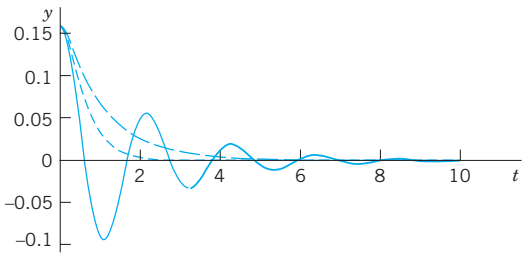


Fig. 40. The three solutions in Example 2

This section concerned ***free motions*** of mass–spring systems. Their models are *homogeneous* linear ODEs. *Nonhomogeneous* linear ODEs will arise as models of **forced motions**, that is, motions under the influence of a "driving force." We shall study them in Sec. 2.8, after we have learned how to solve those ODEs.

PROBLEM SET 2.4

1–10 HARMONIC OSCILLATIONS (UNDAMPED MOTION)

1. **Initial value problem.** Find the harmonic motion (4) that starts from y_0 with initial velocity v_0. Graph or sketch the solutions for $\omega_0 = \pi$, $y_0 = 1$, and various v_0 of your choice on common axes. At what t-values do all these curves intersect? Why?
2. **Frequency.** If a weight of 20 nt (about 4.5 lb) stretches a certain spring by 2 cm, what will the frequency of the corresponding harmonic oscillation be? The period?
3. **Frequency.** How does the frequency of the harmonic oscillation change if we (i) double the mass, (ii) take a spring of twice the modulus? First find qualitative answers by physics, then look at formulas.
4. **Initial velocity.** Could you make a harmonic oscillation move faster by giving the body a greater initial push?
5. **Springs in parallel.** What are the frequencies of vibration of a body of mass $m = 5$ kg (i) on a spring of modulus $k_1 = 20$ nt/m, (ii) on a spring of modulus $k_2 = 45$ nt/m, (iii) on the two springs in parallel? See Fig. 41.

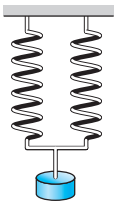


Fig. 41. Parallel springs (Problem 5)

6. **Spring in series.** If a body hangs on a spring s_1 of modulus $k_1 = 8$, which in turn hangs on a spring s_2 of modulus $k_2 = 12$, what is the modulus k of this combination of springs?
7. **Pendulum.** Find the frequency of oscillation of a pendulum of length L (Fig. 42), neglecting air resistance and the weight of the rod, and assuming θ to be so small that $\sin\theta$ practically equals θ.

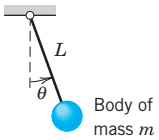


Fig. 42. Pendulum (Problem 7)

8. **Archimedian principle.** This principle states that the buoyancy force equals the weight of the water displaced by the body (partly or totally submerged). The cylindrical buoy of diameter 60 cm in Fig. 43 is floating in water with its axis vertical. When depressed downward in the water and released, it vibrates with period 2 sec. What is its weight?

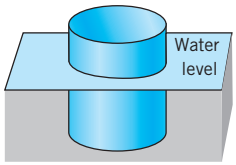


Fig. 43. Buoy (Problem 8)

9. **Vibration of water in a tube.** If 1 liter of water (about 1.06 US quart) is vibrating up and down under the influence of gravitation in a U-shaped tube of diameter 2 cm (Fig. 44), what is the frequency? Neglect friction. First guess.

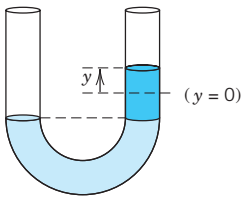


Fig. 44. Tube (Problem 9)

10. **TEAM PROJECT. Harmonic Motions of Similar Models.** The ***unifying power of mathematical methods*** results to a large extent from the fact that different physical (or other) systems may have the same or very similar models. Illustrate this for the following three systems

(a) Pendulum clock. A clock has a 1-meter pendulum. The clock ticks once for each time the pendulum completes a full swing, returning to its original position. How many times a minute does the clock tick?

(b) Flat spring (Fig. 45). The harmonic oscillations of a flat spring with a body attached at one end and horizontally clamped at the other are also governed by (3). Find its motions, assuming that the body weighs 8 nt (about 1.8 lb), the system has its static equilibrium 1 cm below the horizontal line, and we let it start from this position with initial velocity 10 cm/sec.

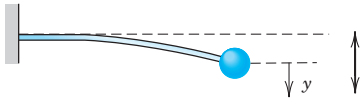


Fig. 45. Flat spring

(c) Torsional vibrations (Fig. 46). Undamped torsional vibrations (rotations back and forth) of a wheel attached to an elastic thin rod or wire are governed by the equation $I_0\theta'' + K\theta = 0$, where θ is the angle measured from the state of equilibrium. Solve this equation for $K/I_0 = 13.69\ \text{sec}^{-2}$, initial angle $30°(= 0.5235\ \text{rad})$ and initial angular velocity $20°\ \text{sec}^{-1}\ (= 0.349\ \text{rad} \cdot \text{sec}^{-1})$.

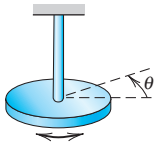


Fig. 46. Torsional vibrations

11–20 DAMPED MOTION

11. Overdamping. Show that for (7) to satisfy initial conditions $y(0) = y_0$ and $v(0) = v_0$ we must have $c_1 = [(1 + \alpha/\beta)y_0 + v_0/\beta]/2$ and $c_2 = [(1 - \alpha/\beta)y_0 - v_0/\beta]/2$.

12. Overdamping. Show that in the overdamped case, the body can pass through $y = 0$ at most once (Fig. 37).

13. Initial value problem. Find the critical motion (8) that starts from y_0 with initial velocity v_0. Graph solution curves for $\alpha = 1$, $y_0 = 1$ and several v_0 such that (i) the curve does not intersect the t-axis, (ii) it intersects it at $t = 1, 2, \ldots, 5$, respectively.

14. Shock absorber. What is the smallest value of the damping constant of a shock absorber in the suspension of a wheel of a car (consisting of a spring and an absorber) that will provide (theoretically) an oscillation-free ride if the mass of the car is 2000 kg and the spring constant equals $4500\ \text{kg/sec}^2$?

15. Frequency. Find an approximation formula for ω^* in terms of ω_0 by applying the binomial theorem in (9) and retaining only the first two terms. How good is the approximation in Example 2, III?

16. Maxima. Show that the maxima of an underdamped motion occur at equidistant t-values and find the distance.

17. Underdamping. Determine the values of t corresponding to the maxima and minima of the oscillation $y(t) = e^{-t}\sin t$. Check your result by graphing $y(t)$.

18. Logarithmic decrement. Show that the ratio of two consecutive maximum amplitudes of a damped oscillation (10) is constant, and the natural logarithm of this ratio called the *logarithmic decrement,* equals $\Delta = 2\pi\alpha/\omega^*$. Find Δ for the solutions of $y'' + 2y' + 5y = 0$.

19. Damping constant. Consider an underdamped motion of a body of mass $m = 0.5$ kg. If the time between two consecutive maxima is 3 sec and the maximum amplitude decreases to $\frac{1}{2}$ its initial value after 10 cycles, what is the damping constant of the system?

20. CAS PROJECT. Transition Between Cases I, II, III. Study this transition in terms of graphs of typical solutions. (Cf. Fig. 47.)

(a) ***Avoiding unnecessary generality is part of good modeling.*** Show that the initial value problems (A) and (B),

$$\text{(A)}\quad y'' + cy' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

(B) the same with different c and $y'(0) = -2$ (instead of 0), will give practically as much information as a problem with other m, k, $y(0)$, $y'(0)$.

(b) ***Consider*** **(A).** Choose suitable values of c, perhaps better ones than in Fig. 47, for the transition from Case III to II and I. Guess c for the curves in the figure.

(c) ***Time to go to rest.*** Theoretically, this time is infinite (why?). Practically, the system is at rest when its motion has become very small, say, less than 0.1% of the initial displacement (this choice being up to us), that is in our case,

$$\text{(11)}\quad |y(t)| < 0.001 \quad \text{for all } t \text{ greater than some } t_1.$$

In engineering constructions, damping can often be varied without too much trouble. Experimenting with your graphs, find empirically a relation between t_1 and c.

(d) ***Solve*** **(A)** ***analytically.*** Give a reason why the solution c of $y(t_2) = -0.001$, with t_2 the solution of $y'(t) = 0$, will give you the best possible c satisfying (11).

(e) Consider (B) empirically as in (a) and (b). What is the main difference between (B) and (A)?

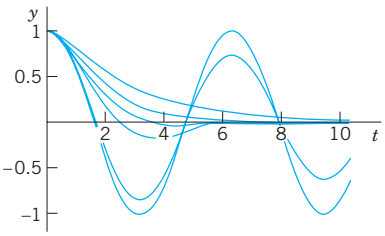


Fig. 47. CAS Project 20

2.5 Euler–Cauchy Equations

Euler–Cauchy equations[4] are ODEs of the form

(1) $$x^2y'' + axy' + by = 0$$

with given constants a and b and unknown function $y(x)$. We substitute

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

into (1). This gives

$$x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$$

and we now see that $y = x^m$ was a rather natural choice because we have obtained a common factor x^m. Dropping it, we have the auxiliary equation $m(m-1) + am + b = 0$ or

(2) $$m^2 + (a-1)m + b = 0. \qquad \text{(Note: } a-1\text{, not } a.\text{)}$$

Hence $y = x^m$ is a solution of (1) if and only if m is a root of (2). The roots of (2) are

(3) $$m_1 = \tfrac{1}{2}(1-a) + \sqrt{\tfrac{1}{4}(1-a)^2 - b}, \qquad m_2 = \tfrac{1}{2}(1-a) - \sqrt{\tfrac{1}{4}(1-a)^2 - b}.$$

Case I. Real different roots m_1 and m_2 give two real solutions

$$y_1(x) = x^{m_1} \qquad \text{and} \qquad y_2(x) = x^{m_2}.$$

These are linearly independent since their quotient is not constant. Hence they constitute a basis of solutions of (1) for all x for which they are real. The corresponding general solution for all these x is

(4) $$y = c_1x^{m_1} + c_2x^{m_2} \qquad (c_1, c_2 \text{ arbitrary}).$$

EXAMPLE 1 General Solution in the Case of Different Real Roots

The Euler–Cauchy equation $x^2y'' + 1.5xy' - 0.5y = 0$ has the auxiliary equation $m^2 + 0.5m - 0.5 = 0$. The roots are 0.5 and -1. Hence a basis of solutions for all positive x is $y_1 = x^{0.5}$ and $y_2 = 1/x$ and gives the general solution

$$y = c_1\sqrt{x} + \frac{c_2}{x} \qquad (x > 0). \quad \blacksquare$$

[4]**LEONHARD EULER (1707–1783)** was an enormously creative Swiss mathematician. He made fundamental contributions to almost all branches of mathematics and its application to physics. His important books on algebra and calculus contain numerous basic results of his own research. The great French mathematician AUGUSTIN LOUIS CAUCHY (1789–1857) is the father of modern analysis. He is the creator of complex analysis and had great influence on ODEs, PDEs, infinite series, elasticity theory, and optics.

Case II. A real double root $m_1 = \frac{1}{2}(1 - a)$ occurs if and only if $b = \frac{1}{4}(a - 1)^2$ because then (2) becomes $[m + \frac{1}{2}(a - 1)]^2$, as can be readily verified. Then a solution is $y_1 = x^{(1-a)/2}$, and (1) is of the form

$$(5) \qquad x^2y'' + axy' + \tfrac{1}{4}(1 - a)^2y = 0 \qquad \text{or} \qquad y'' + \frac{a}{x}y' + \frac{(1 - a)^2}{4x^2}y = 0.$$

A second linearly independent solution can be obtained by the method of reduction of order from Sec. 2.1, as follows. Starting from $y_2 = uy_1$, we obtain for u the expression (9) Sec. 2.1, namely,

$$u = \int U\,dx \qquad \text{where} \qquad U = \frac{1}{y_1^2}\exp\left(-\int p\,dx\right).$$

From (5) in standard form (second ODE) we see that $p = a/x$ (not ax; this is essential!). Hence $\exp\int(-p\,dx) = \exp(-a\ln x) = \exp(\ln x^{-a}) = 1/x^a$. Division by $y_1^2 = x^{1-a}$ gives $U = 1/x$, so that $u = \ln x$ by integration. Thus, $y_2 = uy_1 = y_1 \ln x$, and y_1 and y_2 are linearly independent since their quotient is not constant. The general solution corresponding to this basis is

$$(6) \qquad y = (c_1 + c_2 \ln x)\,x^m, \qquad m = \tfrac{1}{2}(1 - a).$$

EXAMPLE 2 General Solution in the Case of a Double Root

The Euler–Cauchy equation $x^2y'' - 5xy' + 9y = 0$ has the auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive x is

$$y = (c_1 + c_2 \ln x)\,x^3.$$

■

Case III. Complex conjugate roots are of minor practical importance, and we discuss the derivation of real solutions from complex ones just in terms of a typical example.

EXAMPLE 3 Real General Solution in the Case of Complex Roots

The Euler–Cauchy equation $x^2y'' + 0.6xy' + 16.04y = 0$ has the auxiliary equation $m^2 - 0.4m + 16.04 = 0$. The roots are complex conjugate, $m_1 = 0.2 + 4i$ and $m_2 = 0.2 - 4i$, where $i = \sqrt{-1}$. We now use the trick of writing $x = e^{\ln x}$ and obtain

$$x^{m_1} = x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2}e^{(4\ln x)i},$$
$$x^{m_2} = x^{0.2-4i} = x^{0.2}(e^{\ln x})^{-4i} = x^{0.2}e^{-(4\ln x)i}.$$

Next we apply Euler's formula (11) in Sec. 2.2 with $t = 4 \ln x$ to these two formulas. This gives

$$x^{m_1} = x^{0.2}[\cos(4\ln x) + i\sin(4\ln x)],$$
$$x^{m_2} = x^{0.2}[\cos(4\ln x) - i\sin(4\ln x)].$$

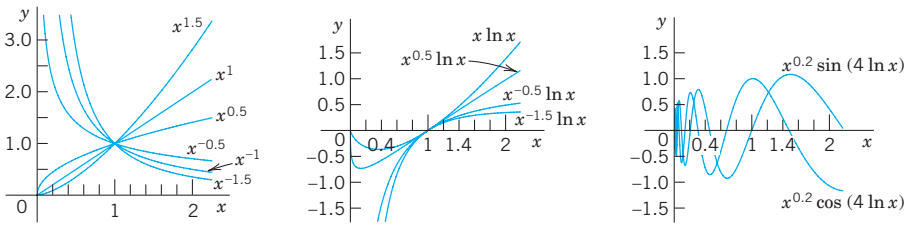
We now add these two formulas, so that the sine drops out, and divide the result by 2. Then we subtract the second formula from the first, so that the cosine drops out, and divide the result by $2i$. This yields

$$x^{0.2}\cos(4\ln x) \qquad \text{and} \qquad x^{0.2}\sin(4\ln x)$$

respectively. By the superposition principle in Sec. 2.2 these are solutions of the Euler–Cauchy equation (1). Since their quotient $\cot(4 \ln x)$ is not constant, they are linearly independent. Hence they form a basis of solutions, and the corresponding real general solution for all positive x is

$$(8) \qquad y = x^{0.2}[A\cos(4\ln x) + B\sin(4\ln x)].$$

Figure 48 shows typical solution curves in the three cases discussed, in particular the real basis functions in Examples 1 and 3.



Case I: Real roots Case II: Double root Case III: Complex roots

Fig. 48. Euler–Cauchy equations

EXAMPLE 4 Boundary Value Problem. Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential $v = v(r)$ between two concentric spheres of radii $r_1 = 5$ cm and $r_2 = 10$ cm kept at potentials $v_1 = 110$ V and $v_2 = 0$, respectively.

Physical Information. $v(r)$ is a solution of the Euler–Cauchy equation $rv'' + 2v' = 0$, where $v' = dv/dr$.

Solution. The auxiliary equation is $m^2 + m = 0$. It has the roots 0 and -1. This gives the general solution $v(r) = c_1 + c_2/r$. From the "boundary conditions" (the potentials on the spheres) we obtain

$$v(5) = c_1 + \frac{c_2}{5} = 110. \qquad v(10) = c_1 + \frac{c_2}{10} = 0.$$

By subtraction, $c_2/10 = 110$, $c_2 = 1100$. From the second equation, $c_1 = -c_2/10 = -110$. *Answer:* $v(r) = -110 + 1100/r$ V. Figure 49 shows that the potential is not a straight line, as it would be for a potential between two parallel plates. For example, on the sphere of radius 7.5 cm it is not $110/2 = 55$ V, but considerably less. (What is it?)

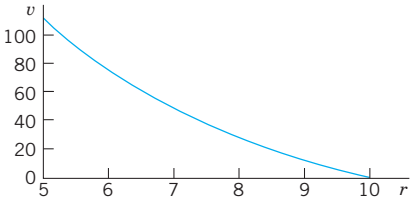


Fig. 49. Potential $v(r)$ in Example 4

PROBLEM SET 2.5

1. Double root. Verify directly by substitution that $x^{(1-a)/2} \ln x$ is a solution of (1) if (2) has a double root, but $x^{m_1} \ln x$ and $x^{m_2} \ln x$ are not solutions of (1) if the roots m_1 and m_2 of (2) are different.

2–11 GENERAL SOLUTION

Find a real general solution. Show the details of your work.

2. $x^2y'' - 20y = 0$

3. $5x^2y'' + 23xy' + 16.2y = 0$

4. $xy'' + 2y' = 0$

5. $4x^2y'' + 5y = 0$

6. $x^2y'' + 0.7xy' - 0.1y = 0$

7. $(x^2D^2 - 4xD + 6I)y = C$

8. $(x^2D^2 - 3xD + 4I)y = 0$

9. $(x^2D^2 - 0.2xD + 0.36I)y = 0$

10. $(x^2D^2 - xD + 5I)y = 0$

11. $(x^2D^2 - 3xD + 10I)y = 0$

12–19 INITIAL VALUE PROBLEM

Solve and graph the solution. Show the details of your work.

12. $x^2y'' - 4xy' + 6y = 0, \quad y(1) = 0.4, \quad y'(1) = 0$

13. $x^2y'' + 3xy' + 0.75y = 0, \quad y(1) = 1, \quad y'(1) = -1.5$

14. $x^2y'' + xy' + 9y = 0, \quad y(1) = 0, \quad y'(1) = 2.5$

15. $x^2y'' + 3xy' + y = 0, \quad y(1) = 3.6, \quad y'(1) = 0.4$

16. $(x^2D^2 - 3xD + 4I)y = 0, \quad y(1) = -\pi, \quad y'(1) = 2\pi$

17. $(x^2D^2 + xD + I)y = 0, \quad y(1) = 1, \quad y'(1) = 1$

18. $(9x^2D^2 + 3xD + I)y = 0, \quad y(1) = 1, \quad y'(1) = 0$

19. $(x^2D^2 - xD - 15I)y = 0, \quad y(1) = 0.1, \quad y'(1) = -4.5$

20. TEAM PROJECT. Double Root

(a) Derive a second linearly independent solution of (1) by reduction of order; but instead of using (9), Sec. 2.1, perform all steps directly for the present ODE (1).

(b) Obtain $x^m \ln x$ by considering the solutions x^m and x^{m+s} of a suitable Euler–Cauchy equation and letting $s \to 0$.

(c) Verify by substitution that $x^m \ln x$, $m = (1 - a)/2$, is a solution in the critical case.

(d) Transform the Euler–Cauchy equation (1) into an ODE with constant coefficients by setting $x = e^t$ $(x > 0)$.

(e) Obtain a second linearly independent solution of the Euler–Cauchy equation in the "critical case" from that of a constant-coefficient ODE.

2.6 Existence and Uniqueness of Solutions. Wronskian

In this section we shall discuss the general theory of homogeneous linear ODEs

(1) $$y'' + p(x)y' + q(x)y = 0$$

with continuous, but otherwise arbitrary, ***variable coefficients*** p and q. This will concern the existence and form of a general solution of (1) as well as the uniqueness of the solution of initial value problems consisting of such an ODE and two initial conditions

(2) $$y(x_0) = K_0, \qquad y'(x_0) = K_1$$

with given x_0, K_0, and K_1.

The two main results will be Theorem 1, stating that such an initial value problem always has a solution which is unique, and Theorem 4, stating that a general solution

(3) $$y = c_1y_1 + c_2y_2 \qquad (c_1, c_2 \text{ arbitrary})$$

includes all solutions. Hence *linear* ODEs with continuous coefficients have no *"singular solutions"* (solutions not obtainable from a general solution).

Clearly, no such theory was needed for constant-coefficient or Euler–Cauchy equations because everything resulted explicitly from our calculations.

Central to our present discussion is the following theorem.

THEOREM 1

Existence and Uniqueness Theorem for Initial Value Problems

If $p(x)$ and $q(x)$ are continuous functions on some open interval I (see Sec. 1.1) *and x_0 is in I, then the initial value problem consisting of* (1) *and* (2) *has a unique solution $y(x)$ on the interval I.*

The proof of existence uses the same prerequisites as the existence proof in Sec. 1.7 and will not be presented here; it can be found in Ref. [A11] listed in App. 1. Uniqueness proofs are usually simpler than existence proofs. But for Theorem 1, even the uniqueness proof is long, and we give it as an additional proof in App. 4.

Linear Independence of Solutions

Remember from Sec. 2.1 that a general solution on an open interval I is made up from a **basis** y_1, y_2 on I, that is, from a pair of linearly independent solutions on I. Here we call y_1, y_2 **linearly independent** on I if the equation

$$k_1y_1(x) + k_2y_2(x) = 0 \quad \text{on } I \qquad \text{implies} \qquad k_1 = 0, \quad k_2 = 0. \tag{4}$$

We call y_1, y_2 **linearly dependent** on I if this equation also holds for constants k_1, k_2 not both 0. In this case, and only in this case, y_1 and y_2 are proportional on I, that is (see Sec. 2.1),

$$\text{(a)} \quad y_1 = ky_2 \qquad \text{or} \qquad \text{(b)} \quad y_2 = ly_1 \qquad \text{for all on } I. \tag{5}$$

For our discussion the following criterion of linear independence and dependence of solutions will be helpful.

THEOREM 2

Linear Dependence and Independence of Solutions

Let the ODE (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval* I. *Then two solutions* y_1 *and* y_2 *of* (1) *on* I *are linearly dependent on* I *if and only if their* "**Wronskian**"

$$W(y_1, y_2) = y_1y_2' - y_2y_1' \tag{6}$$

is 0 *at some* x_0 *in* I. *Furthermore, if* $W = 0$ *at an* $x = x_0$ *in* I, *then* $W = 0$ *on* I; *hence, if there is an* x_1 *in* I *at which* W *is not* 0, *then* y_1, y_2 *are linearly independent on* I.

PROOF **(a)** Let y_1 and y_2 be linearly dependent on I. Then (5a) or (5b) holds on I. If (5a) holds, then

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = ky_2y_2' - y_2ky_2' = 0.$$

Similarly if (5b) holds.

(b) Conversely, we let $W(y_1, y_2) = 0$ for some $x = x_0$ and show that this implies linear dependence of y_1 and y_2 on I. We consider the linear system of equations in the unknowns k_1, k_2

$$\begin{aligned} k_1y_1(x_0) + k_2y_2(x_0) &= 0 \\ k_1y_1'(x_0) + k_2y_2'(x_0) &= 0. \end{aligned} \tag{7}$$

To eliminate k_2, multiply the first equation by y_2' and the second by $-y_2$ and add the resulting equations. This gives

$$k_1 y_1(x_0) y_2'(x_0) - k_1 y_1'(x_0) y_2(x_0) = k_1 W(y_1(x_0), y_2(x_0)) = 0.$$

Similarly, to eliminate k_1, multiply the first equation by $-y_1'$ and the second by y_1 and add the resulting equations. This gives

$$k_2 W(y_1(x_0), y_2(x_0)) = 0.$$

If W were not 0 at x_0, we could divide by W and conclude that $k_1 = k_2 = 0$. Since W is 0, division is not possible, and the system has a solution for which k_1 and k_2 are not both 0. Using *these numbers* k_1, k_2, we introduce the function

$$y(x) = k_1 y_1(x) + k_2 y_2(x).$$

Since (1) is homogeneous linear, Fundamental Theorem 1 in Sec. 2.1 (the superposition principle) implies that this function is a solution of (1) on I. From (7) we see that it satisfies the initial conditions $y(x_0) = 0, y'(x_0) = 0$. Now another solution of (1) satisfying the same initial conditions is $y^* \equiv 0$. Since the coefficients p and q of (1) are continuous, Theorem 1 applies and gives uniqueness, that is, $y \equiv y^*$, written out

$$k_1 y_1 + k_2 y_2 \equiv 0 \qquad \text{on } I.$$

Now since k_1 and k_2 are not both zero, this means linear dependence of y_1, y_2 on I.

(c) We prove the last statement of the theorem. If $W(x_0) = 0$ at an x_0 in I, we have linear dependence of y_1, y_2 on I by part (b), hence $W \equiv 0$ by part (a) of this proof. Hence in the case of linear dependence it cannot happen that $W(x_1) \neq 0$ at an x_1 in I. If it does happen, it thus implies linear independence as claimed. ■

For calculations, the following formulas are often simpler than (6).

(6*) $$W(y_1, y_2) = (a) \quad \left(\frac{y_2}{y_1}\right)' y_1^2 \quad (y_1 \neq 0) \qquad \text{or} \qquad (b) \quad -\left(\frac{y_1}{y_2}\right)' y_2^2 \quad (y_2 \neq 0).$$

These formulas follow from the quotient rule of differentiation.

Remark. Determinants. Students familiar with second-order determinants may have noticed that

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

This determinant is called the *Wronski determinant*[5] or, briefly, the **Wronskian**, of two solutions y_1 and y_2 of (1), as has already been mentioned in (6). Note that its four entries occupy the same positions as in the linear system (7).

[5]Introduced by WRONSKI (JOSEF MARIA HÖNE, 1776–1853), Polish mathematician.

EXAMPLE 1 **Illustration of Theorem 2**

The functions $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of $y'' + \omega^2 y = 0$. Their Wronskian is

$$W(\cos \omega x, \sin \omega x) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = y_1 y_2' - y_2 y_1' = \omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega.$$

Theorem 2 shows that these solutions are linearly independent if and only if $\omega \neq 0$. Of course, we can see this directly from the quotient $y_2/y_1 = \tan \omega x$. For $\omega = 0$ we have $y_2 = 0$, which implies linear dependence (why?). ■

EXAMPLE 2 **Illustration of Theorem 2 for a Double Root**

A general solution of $y'' - 2y' + y = 0$ on any interval is $y = (c_1 + c_2 x)e^x$. (Verify!). The corresponding Wronskian is not 0, which shows linear independence of e^x and xe^x on any interval. Namely,

$$W(x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0.$$ ■

A General Solution of (1) Includes All Solutions

This will be our second main result, as announced at the beginning. Let us start with existence.

THEOREM 3

Existence of a General Solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then (1) *has a general solution on I.*

PROOF By Theorem 1, the ODE (1) has a solution $y_1(x)$ on I satisfying the initial conditions

$$y_1(x_0) = 1, \qquad y_1'(x_0) = 0$$

and a solution $y_2(x)$ on I satisfying the initial conditions

$$y_2(x_0) = 0, \qquad y_2'(x_0) = 1.$$

The Wronskian of these two solutions has at $x = x_0$ the value

$$W(y_1(0), y_2(0)) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 1.$$

Hence, by Theorem 2, these solutions are linearly independent on I. They form a basis of solutions of (1) on I, and $y = c_1 y_1 + c_2 y_2$ with arbitrary c_1, c_2 is a general solution of (1) on I, whose existence we wanted to prove. ■

We finally show that a general solution is as general as it can possibly be.

THEOREM 4

A General Solution Includes All Solutions

If the ODE (1) *has continuous coefficients* $p(x)$ *and* $q(x)$ *on some open interval* I, *then every solution* $y = Y(x)$ *of* (1) *on* I *is of the form*

$$Y(x) = C_1y_1(x) + C_2y_2(x) \tag{8}$$

where y_1, y_2 *is any basis of solutions of* (1) *on* I *and* C_1, C_2 *are suitable constants.*

Hence (1) *does not have* **singular solutions** *(that is, solutions not obtainable from a general solution).*

PROOF Let $y = Y(x)$ be any solution of (1) on I. Now, by Theorem 3 the ODE (1) has a general solution

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{9}$$

on I. We have to find suitable values of c_1, c_2 such that $y(x) = Y(x)$ on I. We choose any x_0 in I and show first that we can find values of c_1, c_2 such that we reach agreement at x_0, that is, $y(x_0) = Y(x_0)$ and $y'(x_0) = Y'(x_0)$. Written out in terms of (9), this becomes

$$\begin{aligned} &\text{(a)} \quad c_1y_1(x_0) + c_2y_2(x_0) = Y(x_0) \\ &\text{(b)} \quad c_1y_1'(x_0) + c_2y_2'(x_0) = Y'(x_0). \end{aligned} \tag{10}$$

We determine the unknowns c_1 and c_2. To eliminate c_2, we multiply (10a) by $y_2'(x_0)$ and (10b) by $-y_2(x_0)$ and add the resulting equations. This gives an equation for c_1. Then we multiply (10a) by $-y_1'(x_0)$ and (10b) by $y_1(x_0)$ and add the resulting equations. This gives an equation for c_2. These new equations are as follows, where we take the values of $y_1, y_1', y_2, y_2', Y, Y'$ at x_0.

$$\begin{aligned} c_1(y_1y_2' - y_2y_1') &= c_1W(y_1, y_2) = Yy_2' - y_2Y' \\ c_2(y_1y_2' - y_2y_1') &= c_2W(y_1, y_2) = y_1Y' - Yy_1'. \end{aligned}$$

Since y_1, y_2 is a basis, the Wronskian W in these equations is not 0, and we can solve for c_1 and c_2. We call the (unique) solution $c_1 = C_1, c_2 = C_2$. By substituting it into (9) we obtain from (9) the particular solution

$$y^*(x) = C_1y_1(x) + C_2y_2(x).$$

Now since C_1, C_2 is a solution of (10), we see from (10) that

$$y^*(x_0) = Y(x_0), \qquad y^{*\prime}(x_0) = Y'(x_0).$$

From the uniqueness stated in Theorem 1 this implies that y^* and Y must be equal everywhere on I, and the proof is complete. ■

Reflecting on this section, we note that homogeneous linear ODEs with continuous variable coefficients have a conceptually and structurally rather transparent existence and uniqueness theory of solutions. Important in itself, this theory will also provide the foundation for our study of nonhomogeneous linear ODEs, whose theory and engineering applications form the content of the remaining four sections of this chapter.

PROBLEM SET 2.6

1. Derive (6*) from (6).

2–8 BASIS OF SOLUTIONS. WRONSKIAN

Find the Wronskian. Show linear independence by using quotients and confirm it by Theorem 2.

2. $e^{4.0x}, e^{-1.5x}$

3. $e^{-0.4x}, e^{-2.6x}$

4. $x, 1/x$

5. x^3, x^2

6. $e^{-x}\cos \omega x, e^{-x}\sin \omega x$

7. $\cosh ax, \sinh ax$

8. $x^k \cos(\ln x), x^k \sin(\ln x)$

9–15 ODE FOR GIVEN BASIS. WRONSKIAN. IVP

(a) Find a second-order homogeneous linear ODE for which the given functions are solutions. **(b)** Show linear independence by the Wronskian. **(c)** Solve the initial value problem.

9. $\cos 5x, \sin 5x, \quad y(0) = 3, \quad y'(0) = -5$

10. $x^{m_1}, x^{m_2}, \quad y(1) = -2, \quad y'(1) = 2m_1 - 4m_2$

11. $e^{-2.5x}\cos 0.3x, e^{-2.5x}\sin 0.3x, \quad y(0) = 3, \quad y'(0) = -7.5$

12. $x^2, x^2 \ln x, \quad y(1) = 4, \quad y'(1) = 6$

13. $1, e^{-2x}, \quad y(0) = 1, \quad y'(0) = -1$

14. $e^{-kx}\cos \pi x, e^{-kx}\sin \pi x, \quad y(0) = 1, \quad y'(0) = -k - \pi$

15. $\cosh 1.8x, \sinh 1.8x, \quad y(0) = 14.20, \quad y'(0) = 16.38$

16. TEAM PROJECT. Consequences of the Present Theory. This concerns some noteworthy general properties of solutions. Assume that the coefficients p and q of the ODE (1) are continuous on some open interval I, to which the subsequent statements refer.

(a) Solve $y'' - y = 0$ (a) by exponential functions, (b) by hyperbolic functions. How are the constants in the corresponding general solutions related?

(b) Prove that the solutions of a basis cannot be 0 at the same point.

(c) Prove that the solutions of a basis cannot have a maximum or minimum at the same point.

(d) Why is it likely that formulas of the form (6*) should exist?

(e) Sketch $y_1(x) = x^3$ if $x \geqq 0$ and 0 if $x < 0$, $y_2(x) = 0$ if $x \geqq 0$ and x^3 if $x < 0$. Show linear independence on $-1 < x < 1$. What is their Wronskian? What Euler–Cauchy equation do y_1, y_2 satisfy? Is there a contradiction to Theorem 2?

(f) Prove **Abel's formula**[6]

$$W(y_1(x), y_2(x)) = c \exp\left[-\int_{x_0}^{x} p(t)\,dt\right]$$

where $c = W(y_1(x_0), y_2(x_0))$. Apply it to Prob. 6. *Hint:* Write (1) for y_1 and for y_2. Eliminate q algebraically from these two ODEs, obtaining a first-order linear ODE. Solve it.

2.7 Nonhomogeneous ODEs

We now advance from homogeneous to nonhomogeneous linear ODEs.

Consider the second-order nonhomogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

where $r(x) \not\equiv 0$. We shall see that a "general solution" of (1) is the sum of a general solution of the corresponding homogeneous ODE

[6]**NIELS HENRIK ABEL** (1802–1829), Norwegian mathematician.

(2) $$y'' + p(x)y' + q(x)y = 0$$

and a "particular solution" of (1). These two new terms "general solution of (1)" and "particular solution of (1)" are defined as follows.

DEFINITION

General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

(3) $$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

Our task is now twofold, first to justify these definitions and then to develop a method for finding a solution y_p of (1).

Accordingly, we first show that a general solution as just defined satisfies (1) and that the solutions of (1) and (2) are related in a very simple way.

THEOREM 1

Relations of Solutions of (1) to Those of (2)

(a) *The sum of a solution y of* (1) *on some open interval I and a solution $\tilde{y}$ of* (2) *on I is a solution of* (1) *on I. In particular,* (3) *is a solution of* (1) *on I.*

(b) *The difference of two solutions of* (1) *on I is a solution of* (2) *on I.*

PROOF **(a)** Let $L[y]$ denote the left side of (1). Then for any solutions y of (1) and $\tilde{y}$ of (2) on I,

$$L[y + \tilde{y}] = L[y] + L[\tilde{y}] = r + 0 = r.$$

(b) For any solutions y and y^* of (1) on I we have $L[y - y^*] = L[y] - L[y^*] = r - r = 0$. ■

Now for *homogeneous ODEs* (2) we know that general solutions include all solutions. We show that the same is true for nonhomogeneous ODEs (1).

THEOREM 2

A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) *are continuous on some open interval I, then every solution of* (1) *on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution* (3) *of* (1) *on I.*

PROOF Let y^* be any solution of (1) on I and x_0 any x in I. Let (3) be any general solution of (1) on I. This solution exists. Indeed, $y_h = c_1y_1 + c_2y_2$ exists by Theorem 3 in Sec. 2.6

because of the continuity assumption, and y_p exists according to a construction to be shown in Sec. 2.10. Now, by Theorem 1(b) just proved, the difference $Y = y^* - y_p$ is a solution of (2) on I. At x_0 we have

$$Y(x_0) = y^*(x_0) - y_p(x_0). \qquad Y'(x_0) = y^{*\prime}(x_0) - y_p'(x_0).$$

Theorem 1 in Sec. 2.6 implies that for these conditions, as for any other initial conditions in I, there exists a unique particular solution of (2) obtained by assigning suitable values to c_1, c_2 in y_h. From this and $y^* = Y + y_p$ the statement follows. ■

Method of Undetermined Coefficients

Our discussion suggests the following. *To solve the nonhomogeneous ODE* (1) *or an initial value problem for* (1), *we have to solve the homogeneous ODE* (2) *and find any solution* y_p *of* (1), so that we obtain a general solution (3) of (1).

How can we find a solution y_p of (1)? One method is the so-called **method of undetermined coefficients**. It is much simpler than another, more general, method (given in Sec. 2.10). Since it applies to models of vibrational systems and electric circuits to be shown in the next two sections, it is frequently used in engineering.

More precisely, the method of undetermined coefficients is suitable for linear ODEs with ***constant coefficients a and b***

$$\textbf{(4)} \qquad y'' + ay' + by = r(x)$$

when $r(x)$ is an exponential function, a power of x, a cosine or sine, or sums or products of such functions. These functions have derivatives similar to $r(x)$ itself. This gives the idea. We choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. Table 2.1 on p. 82 shows the choice of y_p for practically important forms of $r(x)$. Corresponding rules are as follows.

Choice Rules for the Method of Undetermined Coefficients

(a) Basic Rule. *If* $r(x)$ *in* (4) *is one of the functions in the first column in Table* 2.1, *choose* y_p *in the same line and determine its undetermined coefficients by substituting* y_p *and its derivatives into* (4).

(b) Modification Rule. *If a term in your choice for* y_p *happens to be a solution of the homogeneous ODE corresponding to* (4), *multiply this term by* x (*or by* x^2 *if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE*).

(c) Sum Rule. *If* $r(x)$ *is a sum of functions in the first column of Table* 2.1, *choose for* y_p *the sum of the functions in the corresponding lines of the second column.*

The Basic Rule applies when $r(x)$ is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with $r = r_1$ and $r = r_2$ (and the same left side!) is a solution of (1) with $r = r_1 + r_2$. (Verify!)

The method is self-correcting. A false choice for y_p or one with too few terms will lead to a contradiction. A choice with too many terms will give a correct result, with superfluous coefficients coming out zero.

Let us illustrate Rules (a)–(c) by the typical Examples 1–3.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n\ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

EXAMPLE 1 Application of the Basic Rule (a)

Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5. \tag{5}$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The ODE $y'' + y = 0$ has the general solution

$$y_h = A\cos x + B\sin x.$$

Step 2. Solution y_p of the nonhomogeneous ODE. We first try $y_p = Kx^2$. Then $y_p'' = 2K$. By substitution, $2K + Kx^2 = 0.001x^2$. For this to hold for all x, the coefficient of each power of x (x^2 and x^0) must be the same on both sides; thus $K = 0.001$ and $2K = 0$, a contradiction.

The second line in Table 2.1 suggests the choice

$$y_p = K_2x^2 + K_1x + K_0. \qquad \text{Then} \qquad y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2.$$

Equating the coefficients of x^2, x, x^0 on both sides, we have $K_2 = 0.001$, $K_1 = 0$, $2K_2 + K_0 = 0$. Hence $K_0 = -2K_2 = -0.002$. This gives $y_p = 0.001x^2 - 0.002$, and

$$y = y_h + y_p = A\cos x + B\sin x + 0.001x^2 - 0.002.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A\sin x + B\cos x + 0.002x \qquad \text{and} \qquad y'(0) = B = 1.5.$$

This gives the answer (Fig. 50)

$$y = 0.002\cos x + 1.5\sin x + 0.001x^2 - 0.002.$$

Figure 50 shows y as well as the quadratic parabola y_p about which y is oscillating, practically like a sine curve since the cosine term is smaller by a factor of about $1/1000$. ■

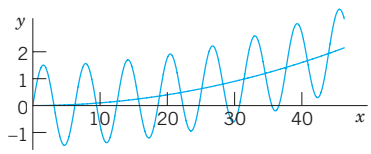


Fig. 50. Solution in Example 1

EXAMPLE 2 Application of the Modification Rule (b)

Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{6}$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}.$$

Step 2. Solution y_p of the nonhomogeneous ODE. The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a *double root* of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2e^{-1.5x}. \qquad \text{Then} \qquad y_p' = C(2x - 1.5x^2)e^{-1.5x}, \qquad y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}.$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0, 0 = 0, 2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^2e^{-1.5x}$. Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2 x)e^{-1.5x} - 5x^2e^{-1.5x}.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives

$$y' = (c_2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}.$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer (Fig. 51)

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases. ■

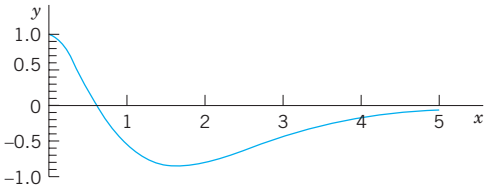


Fig. 51. Solution in Example 2

EXAMPLE 3 Application of the Sum Rule (c)

Solve the initial value problem

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \qquad y(0) = 2.78, \qquad y'(0) = -0.43. \tag{7}$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation of the homogeneous ODE is

$$\lambda^2 + 2\lambda + 0.75 = (\lambda + \tfrac{1}{2})(\lambda + \tfrac{3}{2}) = 0$$

which gives the general solution $y_h = c_1e^{-x/2} + c_2e^{-3x/2}$.

Step 2. Particular solution of the nonhomogeneous ODE. We write $y_p = y_{p1} + y_{p2}$ and, following Table 2.1, (C) and (B),

$$y_{p1} = K\cos x + M\sin x \qquad \text{and} \qquad y_{p2} = K_1 x + K_0.$$

Differentiation gives $y'_{p1} = -K\sin x + M\cos x$, $y''_{p1} = -K\cos x - M\sin x$ and $y'_{p2} = 1$, $y''_{p2} = 0$. Substitution of y_{p1} into the ODE in (7) gives, by comparing the cosine and sine terms,

$$-K + 2M + 0.75K = 2, \qquad -M - 2K + 0.75M = -0.25,$$

hence $K = 0$ and $M = 1$. Substituting y_{p2} into the ODE in (7) and comparing the x- and x^0-terms gives

$$0.75K_1 = 0.09, \quad 2K_1 + 0.75K_0 = 0, \qquad \text{thus} \qquad K_1 = 0.12, \quad K_0 = -0.32.$$

Hence a general solution of the ODE in (7) is

$$y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32.$$

Step 3. Solution of the initial value problem. From y, y' and the initial conditions we obtain

$$y(0) = c_1 + c_2 - 0.32 = 2.78, \qquad y'(0) = -\tfrac{1}{2}c_1 - \tfrac{3}{2}c_2 + 1 + 0.12 = -0.4.$$

Hence $c_1 = 3.1$, $c_2 = 0$. This gives the solution of the IVP (Fig. 52)

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32.$$

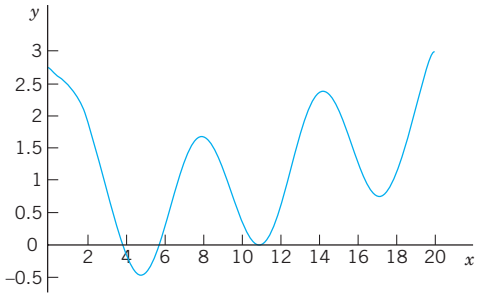


Fig. 52. Solution in Example 3

Stability. The following is important. If (and only if) all the roots of the characteristic equation of the homogeneous ODE $y'' + ay' + by = 0$ in (4) are negative, or have a negative real part, then a general solution y_h of this ODE goes to 0 as $x \to \infty$, so that the "**transient solution**" $y = y_h + y_p$ of (4) approaches the "**steady-state solution**" y_p. In this case the nonhomogeneous ODE and the physical or other system modeled by the ODE are called **stable**; otherwise they are called **unstable**. For instance, the ODE in Example 1 is unstable.

Applications follow in the next two sections.

PROBLEM SET 2.7

1–10 NONHOMOGENEOUS LINEAR ODEs: GENERAL SOLUTION

Find a (real) general solution. State which rule you are using. Show each step of your work.

1. $y'' + 5y' + 4y = 10e^{-3x}$

2. $10y'' + 50y' + 57.6y = \cos x$

3. $y'' + 3y' + 2y = 12x^2$

4. $y'' - 9y = 18\cos \pi x$

5. $y'' + 4y' + 4y = e^{-x}\cos x$

6. $y'' + y' + (\pi^2 + \frac{1}{4})y = e^{-x/2}\sin \pi x$

7. $(D^2 + 2D + \frac{3}{4}I)y = 3e^x + \frac{9}{2}x$
8. $(3D^2 + 27I)y = 3\cos x + \cos 3x$
9. $(D^2 - 16I)y = 9.6e^{4x} + 30e^x$
10. $(D^2 + 2D + I)y = 2x \sin x$

11–18 NONHOMOGENEOUS LINEAR ODEs: IVPs

Solve the initial value problem. State which rule you are using. Show each step of your calculation in detail.

11. $y'' + 3y = 18x^2, \quad y(0) = -3, \quad y'(0) = 0$
12. $y'' + 4y = -12 \sin 2x, \quad y(0) = 1.8, \quad y'(0) = 5.0$
13. $8y'' - 6y' + y = 6 \cosh x, \quad y(0) = 0.2, \quad y'(0) = 0.05$
14. $y'' + 4y' + 4y = e^{-2x} \sin 2x, \quad y(0) = 1, \quad y'(0) = -1.5$
15. $(x^2D^2 - 3xD + 3I)y = 3 \ln x - 4, \quad y(1) = 0, \quad y'(1) = 1; \quad y_p = \ln x$
16. $(D^2 - 2D)y = 6e^{2x} - 4e^{-2x}, \quad y(0) = -1, \quad y'(0) = 6$
17. $(D^2 + 0.2D + 0.26I)y = 1.22e^{0.5x}, \quad y(0) = 3.5, \quad y'(0) = 0.35$
18. $(D^2 + 2D + 10I)y = 17 \sin x - 37 \sin 3x, \quad y(0) = 6.6, \quad y'(0) = -2.2$
19. **CAS PROJECT. Structure of Solutions of Initial Value Problems.** Using the present method, find, graph, and discuss the solutions y of initial value problems of your own choice. Explore effects on solutions caused by changes of initial conditions. Graph $y_p, y, y - y_p$ separately, to see the separate effects. Find a problem in which **(a)** the part of y resulting from y_h decreases to zero, **(b)** increases, **(c)** is not present in the answer y. Study a problem with $y(0) = 0, y'(0) = 0$. Consider a problem in which you need the Modification Rule (a) for a simple root, (b) for a double root. Make sure that your problems cover all three Cases I, II, III (see Sec. 2.2).
20. **TEAM PROJECT. Extensions of the Method of Undetermined Coefficients. (a)** Extend the method to products of the function in Table 2.1, **(b)** Extend the method to Euler–Cauchy equations. Comment on the practical significance of such extensions.

2.8 Modeling: Forced Oscillations. Resonance

In Sec. 2.4 we considered vertical motions of a mass–spring system (vibration of a mass m on an elastic spring, as in Figs. 33 and 53) and modeled it by the ***homogeneous*** linear ODE

$$my'' + cy' + ky = 0. \tag{1}$$

Here $y(t)$ as a function of time t is the displacement of the body of mass m from rest.

The mass–spring system of Sec. 2.4 exhibited only free motion. This means no external forces (outside forces) but only internal forces controlled the motion. The internal forces are forces within the system. They are the force of inertia my'', the damping force cy' (if $c > 0$), and the spring force ky, a restoring force.

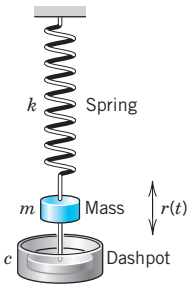


Fig. 53. Mass on a spring

We now extend our model by including an additional force, that is, the external force $r(t)$, on the right. Then we have

$$(2^*) \qquad my'' + cy' + ky = r(t).$$

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the nonhomogeneous ODE

$$\textbf{(2)} \qquad my'' + cy' + ky = F_0 \cos \omega t.$$

Its solution will reveal facts that are fundamental in engineering mathematics and allow us to model resonance.

Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution y_h of the homogeneous ODE (1) plus any solution y_p of (2). To find y_p, we use the method of undetermined coefficients (Sec. 2.7), starting from

$$\textbf{(3)} \qquad y_p(t) = a \cos \omega t + b \sin \omega t.$$

By differentiating this function (chain rule!) we obtain

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$
$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_p, y_p', and y_p'' into (2) and collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cosine terms on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right. This gives the two equations

$$(4) \qquad \begin{aligned} (k - m\omega^2)a + \quad \omega cb \quad &= F_0 \\ -\omega ca \quad + (k - m\omega^2)b \quad &= 0 \end{aligned}$$

for determining the unknown coefficients a and b. This is a linear system. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(k - m\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0 \, (> 0)$ as in Sec. 2.4, then $k = m\omega_0^2$ and we obtain

(5)
$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}.$$

We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

(6)
$$y(t) = y_h(t) + y_p(t).$$

Here y_h is a general solution of the homogeneous ODE (1) and y_p is given by (3) with coefficients (5).

We shall now discuss the behavior of the mechanical system, distinguishing between the two cases $c = 0$ (no damping) and $c > 0$ (damping). These cases will correspond to two basically different types of output.

Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0^2 = k/m$)

(7)
$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t.$$

Here we must assume that $\omega^2 \neq \omega_0^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4]. From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

(8)
$$y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

We see that this output is a ***superposition of two harmonic oscillations*** *of the frequencies just mentioned.*

Resonance. We discuss (7). We see that the maximum amplitude of y_p is (put $\cos \omega t = 1$)

$$(9) \qquad a_0 = \frac{F_0}{k}\rho \qquad \text{where} \qquad \rho = \frac{1}{1 - (\omega/\omega_0)^2}.$$

a_0 depends on ω and ω_0. If $\omega \to \omega_0$, then ρ and a_0 tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance**. ρ is called the **resonance factor** (Fig. 54), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution y_p and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

In the case of resonance the nonhomogeneous ODE (2) becomes

$$(10) \qquad y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t.$$

Then (7) is no longer valid, and, from the Modification Rule in Sec. 2.7, we conclude that a particular solution of (10) is of the form

$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$

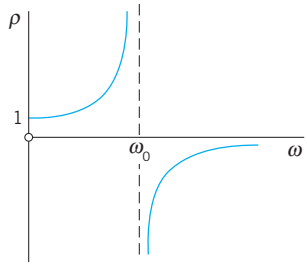


Fig. 54. Resonance factor $\rho(\omega)$

By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 55)

$$\textbf{(11)} \qquad y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

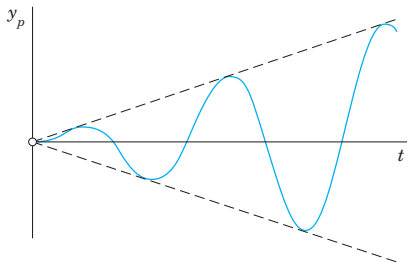


Fig. 55. Particular solution in the case of resonance

We see that, because of the factor t, the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations

that can destroy the system. We shall return to this practical aspect of resonance later in this section.

Beats. Another interesting and highly important type of oscillation is obtained if ω is close to ω_0. Take, for example, the particular solution [see (8)]

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \qquad (\omega \neq \omega_0). \tag{12}$$

Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

Since ω is close to ω_0, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 56, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

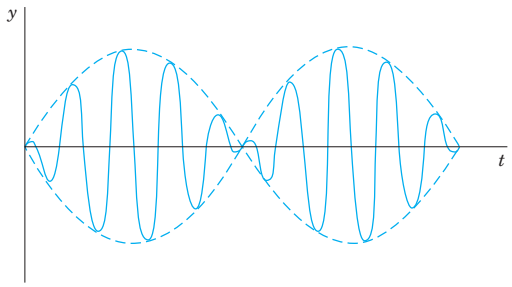


Fig. 56. Forced undamped oscillation when the difference of the input and natural frequencies is small (**"beats"**)

Case 2. Damped Forced Oscillations

If the damping of the mass–spring system is not negligibly small, we have $c > 0$ and a damping term cy' in (1) and (2). Then the general solution y_h of the homogeneous ODE (1) approaches zero as t goes to infinity, as we know from Sec. 2.4. Practically, it is zero after a sufficiently long time. Hence the "**transient solution**" (6) of (2), given by $y = y_h + y_p$, approaches the "**steady-state solution**" y_p. This proves the following.

THEOREM 1

Steady-State Solution

After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force [*see* (2)] *will practically be a harmonic oscillation whose frequency is that of the input.*

Amplitude of the Steady-State Solution. Practical Resonance

Whereas in the undamped case the amplitude of y_p approaches infinity as ω approaches ω_0, this will not happen in the damped case. In this case the amplitude will always be finite. But it may have a maximum for some ω depending on the damping constant c. This may be called **practical resonance**. It is of great importance because if c is not too large, then some input may excite oscillations large enough to damage or even destroy the system. Such cases happened, in particular in earlier times when less was known about resonance. Machines, cars, ships, airplanes, bridges, and high-rising buildings are vibrating mechanical systems, and it is sometimes rather difficult to find constructions that are completely free of undesired resonance effects, caused, for instance, by an engine or by strong winds.

To study the amplitude of y_p as a function of ω, we write (3) in the form

$$y_p(t) = C^* \cos(\omega t - \eta). \tag{13}$$

C^* is called the **amplitude** of y_p and η the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}, \qquad \tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega^2)}. \tag{14}$$

Let us see whether $C^*(\omega)$ has a maximum and, if so, find its location and then its size. We denote the radicand in the second root in C^* by R. Equating the derivative of C^* to zero, we obtain

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2}R^{-3/2}\right)[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2].$$

The expression in the brackets [. . .] is zero if

$$c^2 = 2m^2(\omega_0^2 - \omega^2) \qquad (\omega_0^2 = k/m). \tag{15}$$

By reshuffling terms we have

$$2m^2\omega^2 = 2m^2\omega_0^2 - c^2 = 2mk - c^2.$$

The right side of this equation becomes negative if $c^2 > 2mk$, so that then (15) has no real solution and C^* decreases monotone as ω increases, as the lowest curve in Fig. 57 shows. If c is smaller, $c^2 < 2mk$, then (15) has a real solution $\omega = \omega_{\max}$, where

$$\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m^2}. \tag{15*}$$

From (15*) we see that this solution increases as c decreases and approaches ω_0 as c approaches zero. See also Fig. 57.

The size of $C^*(\omega_{\max})$ is obtained from (14), with $\omega^2 = \omega_{\max}^2$ given by (15*). For this ω^2 we obtain in the second radicand in (14) from (15*)

$$m^2(\omega_0^2 - \omega_{\max}^2)^2 = \frac{c^4}{4m^2} \qquad \text{and} \qquad \omega_{\max}^2 c^2 = \left(\omega_0^2 - \frac{c^2}{2m^2}\right)c^2.$$

The sum of the right sides of these two formulas is

$$(c^4 + 4m^2\omega_0^2c^2 - 2c^4)/(4m^2) = c^2(4m^2\omega_0^2 - c^2)/(4m^2).$$

Substitution into (14) gives

(16)
$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}.$$

We see that $C^*(\omega_{\max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2\omega_0^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as c^2 ($<2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1. Figure 57 shows the **amplification** C^*/F_0 (ratio of the amplitudes of output and input) as a function of ω for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant c.

Figure 58 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.

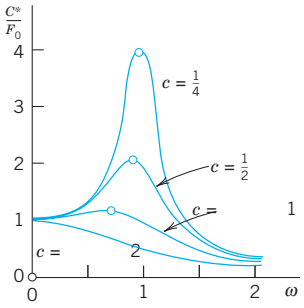


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1$, $k = 1$, and various values of the damping constant c

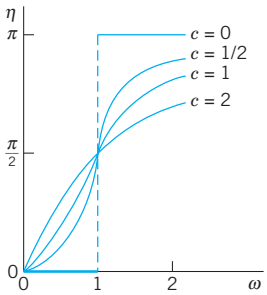


Fig. 58. Phase lag η as a function of ω for $m = 1$, $k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

PROBLEM SET 2.8

1. **WRITING REPORT. Free and Forced Vibrations.** Write a condensed report of 2–3 pages on the most important similarities and differences of free and forced vibrations, with examples of your own. No proofs.

2. **Which of Probs.** 1–18 in Sec. 2.7 (with x = time t) can be models of mass–spring systems with a harmonic oscillation as steady-state solution?

3–7 STEADY-STATE SOLUTIONS

Find the steady-state motion of the mass–spring system modeled by the ODE. Show the details of your work.

3. $y'' + 6y' + 8y = 42.5 \cos 2t$

4. $y'' + 2.5y' + 10y = -13.6 \sin 4t$

5. $(D^2 + D + 4.25I)y = 22.1 \cos 4.5t$

6. $(D^2 + 4D + 3I)y = \cos t + \frac{1}{3}\cos 3t$

7. $(4D^2 + 12D + 9I)y = 225 - 75 \sin 3t$

8–15 TRANSIENT SOLUTIONS

Find the transient motion of the mass–spring system modeled by the ODE. Show the details of your work.

8. $2y'' + 4y' + 6.5y = 4 \sin 1.5t$

9. $y'' + 3y' + 3.25y = 3 \cos t - 1.5 \sin t$

10. $y'' + 16y = 56 \cos 4t$

11. $(D^2 + 2I)y = \cos \sqrt{2}t + \sin\sqrt{2}t$

12. $(D^2 + 2D + 5I)y = 4 \cos t + 8 \sin t$

13. $(D^2 + I)y = \cos \omega t, \omega^2 \neq 1$

14. $(D^2 + I)y = 5e^{-t} \cos t$

15. $(D^2 + 4D + 8I)y = 2 \cos 2t + \sin 2t$

16–20 INITIAL VALUE PROBLEMS

Find the motion of the mass–spring system modeled by the ODE and the initial conditions. Sketch or graph the solution curve. In addition, sketch or graph the curve of $y - y_p$ to see when the system practically reaches the steady state.

16. $y'' + 25y = 24 \sin t, \quad y(0) = 1, \quad y'(0) = 1$

17. $(D^2 + 4I)y = \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t,$
$y(0) = 0, \quad y'(0) = \frac{3}{35}$

18. $(D^2 + 8D + 17I)y = 474.5 \sin 0.5t, \quad y(0) = -5.4,$
$y'(0) = 9.4$

19. $(D^2 + 2D + 2I)y = e^{-t/2}\sin\frac{1}{2}t, \quad y(0) = 0,$
$y'(0) = 1$

20. $(D^2 + 5I)y = \cos \pi t - \sin \pi t, \quad y(0) = 0, \quad y'(0) = 0$

21. Beats. Derive the formula after (12) from (12). Can we have beats in a damped system?

22. Beats. Solve $y'' + 25y = 99 \cos 4.9t, \quad y(0) = 2,$ $y'(0) = 0$. How does the graph of the solution change if you change **(a)** $y(0)$, **(b)** the frequency of the driving force?

23. TEAM EXPERIMENT. Practical Resonance.
(a) Derive, in detail, the crucial formula (16).

(b) By considering dC^*/dc show that $C^*(\omega_{\max})$ increases as c ($\leqq \sqrt{2mk}$) decreases.

(c) Illustrate practical resonance with an ODE of your own in which you vary c, and sketch or graph corresponding curves as in Fig. 57.

(d) Take your ODE with c fixed and an input of two terms, one with frequency close to the practical resonance frequency and the other not. Discuss and sketch or graph the output.

(e) Give other applications (not in the book) in which resonance is important.

24. Gun barrel. Solve $y'' + y = 1 - t^2/\pi^2$ if $0 \leqq t \leqq \pi$ and 0 if $t \to \infty$; here, $y(0) = 0, y'(0) = 0$. This models an undamped system on which a force F acts during some interval of time (see Fig. 59), for instance, the force on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot, which we disregard for simplicity). *Hint:* At π both y and y' must be continuous.

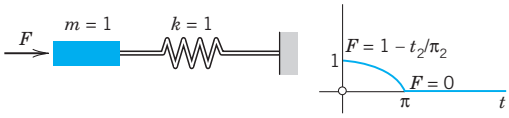


Fig. 59. Problem 24

25. CAS EXPERIMENT. Undamped Vibrations.
(a) Solve the initial value problem $y'' + y = \cos \omega t, \omega^2 \neq 1, y(0) = 0, y'(0) = 0$. Show that the solution can be written

$$y(t) = \frac{2}{1 - \omega^2} \sin [\tfrac{1}{2}(1 + \omega)t] \sin [\tfrac{1}{2}(1 - \omega)t].$$

(b) Experiment with the solution by changing ω to see the change of the curves from those for small ω (>0) to beats, to resonance, and to large values of ω (see Fig. 60).

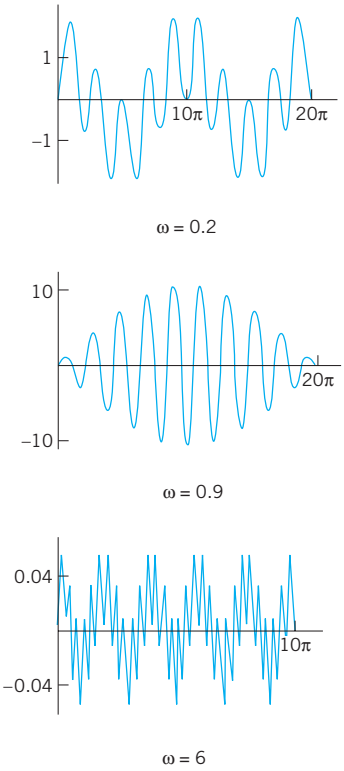


Fig. 60. Typical solution curves in CAS Experiment 25

2.9 Modeling: Electric Circuits

Designing good models is a task the computer cannot do. Hence setting up models has become an important task in modern applied mathematics. The best way to gain experience in successful modeling is to carefully examine the modeling process in various fields and applications. Accordingly, modeling electric circuits will be *profitable for all students,* not just for electrical engineers and computer scientists.

Figure 61 shows an ***RLC*-circuit**, as it occurs as a basic building block of large electric networks in computers and elsewhere. An *RLC*-circuit is obtained from an *RL*-circuit by adding a capacitor. Recall Example 2 on the *RL*-circuit in Sec. 1.5: The model of the *RL*-circuit is $LI' + RI = E(t)$. It was obtained by **KVL** (Kirchhoff's Voltage Law)[7] by equating the voltage drops across the resistor and the inductor to the EMF (electromotive force). Hence we obtain the model of the *RLC*-circuit simply by adding the voltage drop Q/C across the capacitor. Here, C F (farads) is the capacitance of the capacitor. Q coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt}, \qquad \text{equivalently} \qquad Q(t) = \int I(t)\,dt.$$

See also Fig. 62. Assuming a sinusoidal EMF as in Fig. 61, we thus have the model of the *RLC*-circuit

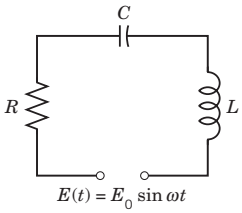


Fig. 61. *RLC*-circuit

Name	Symbol		Notation	Unit	Voltage Drop
Ohm's Resistor		R	Ohm's Resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L\frac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

Fig. 62. Elements in an *RLC*-circuit

[7]GUSTAV ROBERT KIRCHHOFF (1824–1887), German physicist. Later we shall also need **Kirchhoff's Current Law (KCL)**:

At any point of a circuit, the sum of the inflowing currents is equal to the sum of the outflowing currents.

The units of measurement of electrical quantities are named after ANDRÉ MARIE AMPÈRE (1775–1836), French physicist, CHARLES AUGUSTIN DE COULOMB (1736–1806), French physicist and engineer, MICHAEL FARADAY (1791–1867), English physicist, JOSEPH HENRY (1797–1878), American physicist, GEORG SIMON OHM (1789–1854), German physicist, and ALESSANDRO VOLTA (1745–1827), Italian physicist.

$$(1') \qquad LI' + RI + \frac{1}{C}\int I\,dt = E(t) = E_0 \sin \omega t.$$

This is an "integro-differential equation." To get rid of the integral, we differentiate $(1')$ with respect to t, obtaining

$$\textbf{(1)} \qquad LI'' + RI' + \frac{1}{C}I = E'(t) = E_0\omega \cos \omega t.$$

This shows that the current in an *RLC*-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients.

In connection with initial value problems, we shall occasionally use

$$(1'') \qquad LQ'' + RQ'' + \frac{1}{C}Q = E(t),$$

obtained from $(1')$ and $I = Q'$.

Solving the ODE (1) for the Current in an *RLC*-Circuit

A general solution of (1) is the sum $I = I_h + I_p$, where I_h is a general solution of the homogeneous ODE corresponding to (1) and I_p is a particular solution of (1). We first determine I_p by the method of undetermined coefficients, proceeding as in the previous section. We substitute

$$(2) \qquad \begin{aligned} I_p &= a \cos \omega t + b \sin \omega t \\ I_p' &= \omega(-a \sin \omega t + b \cos \omega t) \\ I_p'' &= \omega^2(-a \cos \omega t - b \sin \omega t) \end{aligned}$$

into (1). Then we collect the cosine terms and equate them to $E_0\omega \cos \omega t$ on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$\begin{aligned} L\omega^2(-a) + R\omega b + a/C &= E_0\omega \qquad \text{(Cosine terms)} \\ L\omega^2(-b) + R\omega(-a) + b/C &= 0 \qquad \text{(Sine terms).} \end{aligned}$$

Before solving this system for a and b, we first introduce a combination of L and C, called the **reactance**

$$\textbf{(3)} \qquad S = \omega L - \frac{1}{\omega C}.$$

Dividing the previous two equations by ω, ordering them, and substituting S gives

$$\begin{aligned} -Sa + Rb &= E_0 \\ -Ra - Sb &= 0. \end{aligned}$$

We now eliminate b by multiplying the first equation by S and the second by R, and adding. Then we eliminate a by multiplying the first equation by R and the second by $-S$, and adding. This gives

$$-(S^2 + R^2)a = E_0 S, \qquad (R^2 + S^2)b = E_0 R.$$

We can solve for a and b,

$$a = \frac{-E_0 S}{R^2 + S^2}, \qquad b = \frac{E_0 R}{R^2 + S^2}. \tag{4}$$

Equation (2) with coefficients a and b given by (4) is the desired particular solution I_p of the nonhomogeneous ODE (1) governing the current I in an RLC-circuit with sinusoidal electromotive force.

Using (4), we can write I_p in terms of "physically visible" quantities, namely, amplitude I_0 and phase lag θ of the current behind the EMF, that is,

$$I_p(t) = I_0 \sin(\omega t - \theta) \tag{5}$$

where [see (14) in App. A3.1]

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \qquad \tan\theta = -\frac{a}{b} = \frac{S}{R}.$$

The quantity $\sqrt{R^2 + S^2}$ is called the **impedance**. Our formula shows that the impedance equals the ratio E_0/I_0. This is somewhat analogous to $E/I = R$ (Ohm's law) and, because of this analogy, the impedance is also known as the **apparent resistance**.

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0.$$

We can write these roots in the form $\lambda_1 = -\alpha + \beta$ and $\lambda_2 = -\alpha - \beta$, where

$$\alpha = \frac{R}{2L}, \qquad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}.$$

Now in an actual circuit, R is never zero (hence $R > 0$). From this it follows that I_h approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time. Hence the transient current $I = I_h + I_p$ tends to the steady-state current I_p, and after some time the output will practically be a harmonic oscillation, which is given by (5) and whose frequency is that of the input (of the electromotive force).

EXAMPLE 1 ***RLC*-Circuit**

Find the current $I(t)$ in an RLC-circuit with $R = 11\ \Omega$ (ohms), $L = 0.1$ H (henry), $C = 10^{-2}$ F (farad), which is connected to a source of EMF $E(t) = 110 \sin (60 \cdot 2\pi t) = 110 \sin 377\,t$ (hence 60 Hz = 60 cycles/sec, the usual in the U.S. and Canada; in Europe it would be 220 V and 50 Hz). Assume that current and capacitor charge are 0 when $t = 0$.

Solution. ***Step 1. General solution of the homogeneous ODE.*** Substituting R, L, C and the derivative $E'(t)$ into (1), we obtain

$$0.1I'' + 11I' + 100I = 110 \cdot 377 \cos 377t.$$

Hence the homogeneous ODE is $0.1I'' + 11I' + 100I = 0$. Its characteristic equation is

$$0.1\lambda^2 + 11\lambda + 100 = 0.$$

The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$

Step 2. Particular solution I_p of (1). We calculate the reactance $S = 37.7 - 0.3 = 37.4$ and the steady-state current

$$I_p(t) = a \cos 377t + b \sin 377t$$

with coefficients obtained from (4) (and rounded)

$$a = \frac{-110 \cdot 37.4}{11^2 + 37.4^2} = -2.71, \qquad b = \frac{110 \cdot 11}{11^2 + 37.4^2} = 0.796.$$

Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

$$I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t. \tag{6}$$

Step 3. Particular solution satisfying the initial conditions. **How to use $Q(0) = 0$?** We finally determine c_1 and c_2 from the in initial conditions $I(0) = 0$ and $Q(0) = 0$. From the first condition and (6) we have

$$I(0) = c_1 + c_2 - 2.71 = 0, \qquad \text{hence} \qquad c_2 = 2.71 - c_1. \tag{7}$$

We turn to $Q(0) = 0$. The integral in $(1')$ equals $\int I\,dt = Q(t)$; see near the beginning of this section. Hence for $t = 0$, Eq. $(1')$ becomes

$$LI'(0) + R \cdot 0 = 0, \qquad \text{so that} \qquad I'(0) = 0.$$

Differentiating (6) and setting $t = 0$, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.796 \cdot 377 = 0, \qquad \text{hence by (7),} \qquad -10c_1 = 100(2.71 - c_1) - 300.1.$$

The solution of this and (7) is $c_1 = -0.323$, $c_2 = 3.033$. Hence the answer is

$$I(t) = -0.323e^{-10t} + 3.033e^{-100t} - 2.71 \cos 377t + 0.796 \sin 377t.$$

You may get slightly different values depending on the rounding. Figure 63 shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly. Thus after a very short time the current will practically execute harmonic oscillations of the input frequency 60 Hz = 60 cycles/sec. Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.824 \sin (377t - 1.29).$$

■

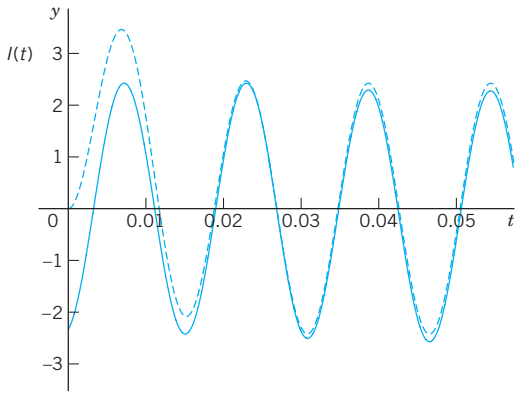


Fig. 63. Transient (upper curve) and steady-state currents in Example 1

Analogy of Electrical and Mechanical Quantities

Entirely different physical or other systems may have the same mathematical model. For instance, we have seen this from the various applications of the ODE $y' = ky$ in Chap. 1. Another impressive demonstration of this ***unifying power of mathematics*** is given by the ODE (1) for an electric *RLC*-circuit and the ODE (2) in the last section for a mass–spring system. Both equations

$$LI'' + RI' + \frac{1}{C}I = E_0\omega \cos \omega t \qquad \text{and} \qquad my'' + cy' + ky = F_0 \cos \omega t$$

are of the same form. Table 2.2 shows the analogy between the various quantities involved. The inductance L corresponds to the mass m and, indeed, an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance R corresponds to the damping constant c, and a resistor causes loss of energy, just as a damping dashpot does. And so on.

This analogy is ***strictly quantitative*** in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

The ***practical importance*** of this analogy is almost obvious. The analogy may be used for constructing an "electrical model" of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.

Table 2.2 Analogy of Electrical and Mechanical Quantities

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega \cos \omega t$ of electromotive force	Driving force $F_0 \cos \omega t$
Current $I(t)$	Displacement $y(t)$

Related to this analogy are **transducers**, devices that convert changes in a mechanical quantity (for instance, in a displacement) into changes in an electrical quantity that can be monitored; see Ref. [GenRef11] in App. 1.

PROBLEM SET 2.9

1–6 *RLC*-CIRCUITS: SPECIAL CASES

1. *RC*-Circuit. Model the *RC*-circuit in Fig. 64. Find the current due to a constant E.

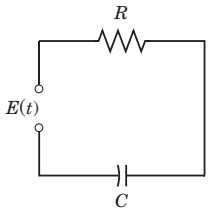


Fig. 64. *RC*-circuit

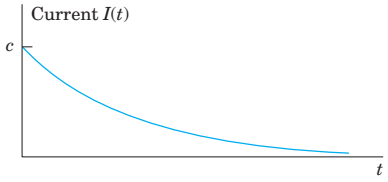


Fig. 65. Current 1 in Problem 1

2. *RC*-Circuit. Solve Prob. 1 when $E = E_0 \sin \omega t$ and R, C, E_0, and ω are arbitrary.

3. *RL*-Circuit. Model the *RL*-circuit in Fig. 66. Find a general solution when R, L, E are any constants. Graph or sketch solutions when $L = 0.25$ H, $R = 10\ \Omega$, and $E = 48$ V.

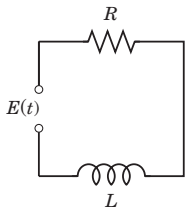


Fig. 66. *RL*-circuit

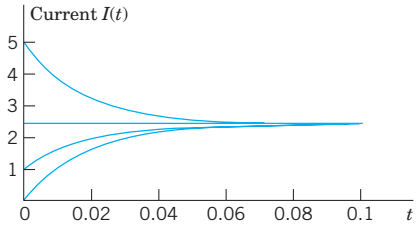


Fig. 67. Currents in Problem 3

4. *RL*-Circuit. Solve Prob. 3 when $E = E_0 \sin \omega t$ and R, L, E_0, and are arbitrary. Sketch a typical solution.

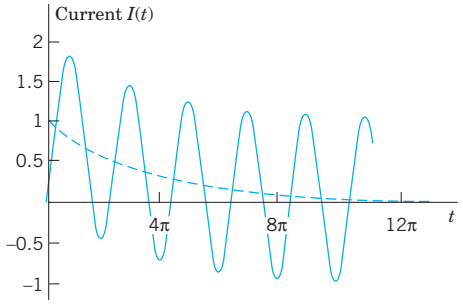


Fig. 68. Typical current $I = e^{-0.1t} + \sin(t - \frac{1}{4}\pi)$ in Problem 4

5. *LC*-Circuit. This is an *RLC*-circuit with negligibly small R (analog of an undamped mass–spring system). Find the current when $L = 0.5$ H, $C = 0.005$ F, and $E = \sin t$ V, assuming zero initial current and charge.

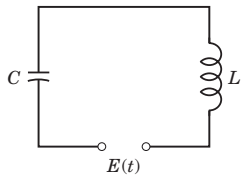


Fig. 69. *LC*-circuit

6. *LC*-Circuit. Find the current when $L = 0.5$ H, $C = 0.005$ F, $E = 2t^2$ V, and initial current and charge zero.

7–18 GENERAL *RLC*-CIRCUITS

7. Tuning. In tuning a stereo system to a radio station, we adjust the tuning control (turn a knob) that changes C (or perhaps L) in an *RLC*-circuit so that the amplitude of the steady-state current (5) becomes maximum. For what C will this happen?

8–14 Find the **steady-state current** in the *RLC*-circuit in Fig. 61 for the given data. Show the details of your work.

8. $R = 4\ \Omega$, $L = 0.5$ H, $C = 0.1$ F, $E = 500 \sin 2t$ V

9. $R = 4\ \Omega$, $L = 0.1$ H, $C = 0.05$ F, $E = 110$ V

10. $R = 2\ \Omega$, $L = 1$ H, $C = \frac{1}{20}$ F, $E = 157 \sin 3t$ V

11. $R = 12\ \Omega$, $L = 0.4$ H, $C = \frac{1}{80}$ F, $E = 220 \sin 10t$ V

12. $R = 0.2\ \Omega$, $L = 0.1$ H, $C = 2$ F, $E = 220 \sin 314t$ V

13. $R = 12$, $L = 1.2$ H, $C = \frac{20}{3} \cdot 10^{-3}$ F, $E = 12{,}000 \sin 25t$ V

14. Prove the claim in the text that if $R \neq 0$ (hence $R > 0$), then the transient current approaches I_p as $t \to \infty$.

15. Cases of damping. What are the conditions for an *RLC*-circuit to be (I) overdamped, (II) critically damped, (III) underdamped? What is the critical resistance R_{crit} (the analog of the critical damping constant $2\sqrt{mk}$)?

16–18 Solve the **initial value problem** for the *RLC*-circuit in Fig. 61 with the given data, assuming zero initial current and charge. Graph or sketch the solution. Show the details of your work.

16. $R = 8\ \Omega$, $L = 0.2$ H, $C = 12.5 \cdot 10^{-3}$ F, $E = 100 \sin 10t$ V

17. $R = 6\ \Omega$, $L = 1$ H, $C = 0.04$ F, $E = 600 (\cos t + 4 \sin t)$ V

18. $R = 18\ \Omega$, $L = 1$ H, $C = 12.5 \cdot 10^{-3}$ F, $E = 820 \cos 10t$ V

19. WRITING REPORT. Mechanic-Electric Analogy. Explain Table 2.2 in a 1–2 page report with examples, e.g., the analog (with $L = 1$ H) of a mass–spring system of mass 5 kg, damping constant 10 kg/sec, spring constant 60 kg/sec^2, and driving force 220 cos 10t kg/sec.

20. Complex Solution Method. Solve $L\tilde{I}'' + R\tilde{I}' + \tilde{I}/C = E_0 e^{i\omega t}$, $i = \sqrt{-1}$, by substituting $I_p = Ke^{i\omega t}$ (K unknown) and its derivatives and taking the real part I_p of the solution $\tilde{I}_p$. Show agreement with (2), (4). *Hint*: Use (11) $e^{i\omega t} = \cos \omega t + i \sin \omega t$; cf. Sec. 2.2, and $i^2 = -1$.

2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs, that is

$$y'' + p(x)y' + q(x)y = r(x). \tag{1}$$

In Sec. 2.6 we have seen that a general solution of (1) is the sum of a general solution y_h of the corresponding homogeneous ODE and any particular solution y_p of (1). To obtain y_p when $r(x)$ is not too complicated, we can often use the *method of undetermined coefficients,* as we have shown in Sec. 2.7 and applied to basic engineering models in Secs. 2.8 and 2.9.

However, since this method is restricted to functions $r(x)$ whose derivatives are of a form similar to $r(x)$ itself (powers, exponential functions, etc.), it is desirable to have a method valid for more general ODEs (1), which we shall now develop. It is called the **method of variation of parameters** and is credited to Lagrange (Sec. 2.1). Here p, q, r in (1) may be variable (given functions of x), but we assume that they are continuous on some open interval I.

Lagrange's method gives a particular solution y_p of (1) on I in the form

$$y_p(x) = -y_1 \int \frac{y_2 r}{W}\, dx + y_2 \int \frac{y_1 r}{W}\, dx \tag{2}$$

where y_1, y_2 form a basis of solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

on I, and W is the Wronskian of y_1, y_2,

$$W = y_1 y_2' - y_2 y_1' \qquad \text{(see Sec. 2.6).} \tag{4}$$

CAUTION! The solution formula (2) is obtained under the assumption that the ODE is written in standard form, with y'' as the first term as shown in (1). If it starts with $f(x)y''$, divide first by $f(x)$.

The integration in (2) may often cause difficulties, and so may the determination of y_1, y_2 if (1) has variable coefficients. If you have a choice, use the previous method. It is simpler. Before deriving (2) let us work an example for which you *do need* the new method. (Try otherwise.)

EXAMPLE 1 Method of Variation of Parameters

Solve the nonhomogeneous ODE

$$y'' + y = \sec x = \frac{1}{\cos x}.$$

Solution. A basis of solutions of the homogeneous ODE on any interval is $y_1 = \cos x$, $y_2 = \sin x$. This gives the Wronskian

$$W(y_1, y_2) = \cos x \cos x - \sin x\,(-\sin x) = 1.$$

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

$$\begin{aligned} y_p &= -\cos x \int \sin x \sec x\, dx + \sin x \int \cos x \sec x\, dx \\ &= \cos x \ln |\cos x| + x \sin x \end{aligned} \qquad \text{(Fig. 70)}$$

Figure 70 shows y_p and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of y_p. (Recall from Sec. 2.8 that we have seen $x \sin x$ in connection with resonance, except for notation.) From y_p and the general solution $y_h = c_1 y_1 + c_2 y_2$ of the homogeneous ODE we obtain the a*nswer*

$$y = y_h + y_p = (c_1 + \ln |\cos x|) \cos x + (c_2 + x) \sin x.$$

Had we included integration constants $-c_1, c_2$ in (2), then (2) would have given the additional $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case. ■

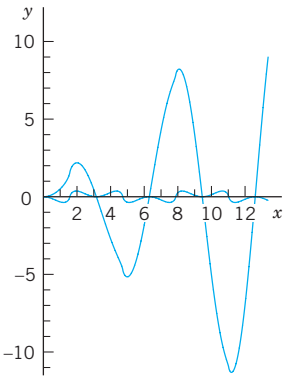


Fig. 70. Particular solution y_p and its first term in Example 1

Idea of the Method. Derivation of (2)

What idea did Lagrange have? What gave the method the name? Where do we use the continuity assumptions?

The idea is to start from a general solution

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

of the homogeneous ODE (3) on an open interval I and to replace the constants ("the parameters") c_1 and c_2 by functions $u(x)$ and $v(x)$; this suggests the name of the method. We shall determine u and v so that the resulting function

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \tag{5}$$

is a particular solution of the nonhomogeneous ODE (1). Note that y_h exists by Theorem 3 in Sec. 2.6 because of the continuity of p and q on I. (The continuity of r will be used later.)

We determine u and v by substituting (5) and its derivatives into (1). Differentiating (5), we obtain

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'.$$

Now y_p must satisfy (1). This is *one* condition for *two* functions u and v. It seems plausible that we may impose a *second* condition. Indeed, our calculation will show that we can determine u and v such that y_p satisfies (1) and u and v satisfy as a second condition the equation

$$u'y_1 + v'y_2 = 0. \tag{6}$$

This reduces the first derivative y_p' to the simpler form

$$y_p' = uy_1' + vy_2'. \tag{7}$$

Differentiating (7), we obtain

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''. \tag{8}$$

We now substitute y_p and its derivatives according to (5), (7), (8) into (1). Collecting terms in u and terms in v, we obtain

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = r.$$

Since y_1 and y_2 are solutions of the homogeneous ODE (3), this reduces to

$$u'y_1' + v'y_2' = r. \tag{9a}$$

Equation (6) is

$$u'y_1 + v'y_2 = 0. \tag{9b}$$

This is a linear system of two algebraic equations for the unknown functions u' and v'. We can solve it by elimination as follows (or by Cramer's rule in Sec. 7.6). To eliminate v', we multiply (9a) by $-y_2$ and (9b) by y_2' and add, obtaining

$$u'(y_1y_2' - y_2y_1') = -y_2r, \qquad \text{thus} \qquad u'W = -y_2r.$$

Here, W is the Wronskian (4) of y_1, y_2. To eliminate u' we multiply (9a) by y_1, and (9b) by $-y_1'$ and add, obtaining

$$v'(y_1y_2' - y_2y_1') = -y_1r, \qquad \text{thus} \qquad v'W = y_1r.$$

Since y_1, y_2 form a basis, we have $W \neq 0$ (by Theorem 2 in Sec. 2.6) and can divide by W,

$$u' = -\frac{y_2r}{W}, \qquad v' = \frac{y_1r}{W}. \tag{10}$$

By integration,

$$u = -\int \frac{y_2r}{W}\,dx, \qquad v = \int \frac{y_1r}{W}\,dx.$$

These integrals exist because $r(x)$ is continuous. Inserting them into (5) gives (2) and completes the derivation. ■

PROBLEM SET 2.10

1–13 GENERAL SOLUTION

Solve the given nonhomogeneous linear ODE by variation of parameters or undetermined coefficients. Show the details of your work.

1. $y'' + 9y = \sec 3x$
2. $y'' + 9y = \csc 3x$
3. $x^2y'' - 2xy' + 2y = x^3 \sin x$
4. $y'' - 4y' + 5y = e^{2x} \csc x$
5. $y'' + y = \cos x - \sin x$
6. $(D^2 + 6D + 9I)y = 16e^{-3x}/(x^2 + 1)$
7. $(D^2 - 4D + 4I)y = 6e^{2x}/x^4$
8. $(D^2 + 4I)y = \cosh 2x$
9. $(D^2 - 2D + I)y = 35x^{3/2}e^x$
10. $(D^2 + 2D + 2I)y = 4e^{-x} \sec^3 x$
11. $(x^2D^2 - 4xD + 6I)y = 21x^{-4}$
12. $(D^2 - I)y = 1/\cosh x$
13. $(x^2D^2 + xD - 9I)y = 48x^5$
14. **TEAM PROJECT. Comparison of Methods. Invention.** The undetermined-coefficient method should be used whenever possible because it is simpler. Compare it with the present method as follows.

    (a) Solve $y'' + 4y' + 3y = 65 \cos 2x$ by both methods, showing all details, and compare.

    (b) Solve $y'' - 2y' + y = r_1 + r_2$, $r_1 = 35x^{3/2}e^x$ $r_2 = x^2$ by applying each method to a suitable function on the right.

    (c) Experiment to invent an undetermined-coefficient method for nonhomogeneous Euler–Cauchy equations.

CHAPTER 2 REVIEW QUESTIONS AND PROBLEMS

1. Why are linear ODEs preferable to nonlinear ones in modeling?
2. What does an initial value problem of a second-order ODE look like? Why must you have a general solution to solve it?
3. By what methods can you get a general solution of a nonhomogeneous ODE from a general solution of a homogeneous one?
4. Describe applications of ODEs in mechanical systems. What are the electrical analogs of the latter?
5. What is resonance? How can you remove undesirable resonance of a construction, such as a bridge, a ship, or a machine?
6. What do you know about existence and uniqueness of solutions of linear second-order ODEs?

7–18 GENERAL SOLUTION

Find a general solution. Show the details of your calculation.

7. $4y'' + 32y' + 63y = 0$
8. $y'' + y' - 12y = 0$
9. $y'' + 6y' + 34y = 0$
10. $y'' + 0.20y' + 0.17y = 0$
11. $(100D^2 - 160D + 64I)y = 0$
12. $(D^2 + 4\pi D + 4\pi^2 I)y = 0$
13. $(x^2D^2 + 2xD - 12I)y = 0$
14. $(x^2D^2 + xD - 9I)y = 0$
15. $(2D^2 - 3D - 2I)y = 13 - 2x^2$
16. $(D^2 + 2D + 2I)y = 3e^{-x} \cos 2x$
17. $(4D^2 - 12D + 9I)y = 2e^{1.5x}$
18. $yy'' = 2y'^2$

19–22 INITIAL VALUE PROBLEMS

Solve the problem, showing the details of your work. Sketch or graph the solution.

19. $y'' + 16y = 17e^x, \quad y(0) = 6, \quad y'(0) = -2$

20. $y'' - 3y' + 2y = 10 \sin x, \quad y(0) = 1, \quad y'(0) = -6$

21. $(x^2D^2 + xD - I)y = 16x^3, \quad y(1) = -1, \quad y'(1) = 1$

22. $(x^2D^2 + 15xD + 49I)y = 0, \quad y(1) = 2, \quad y'(1) = -11$

23–30 APPLICATIONS

23. Find the steady-state current in the *RLC*-circuit in Fig. 71 when $R = 2\,\text{k}\Omega$ (2000 Ω), $L = 1$ H, $C = 4 \cdot 10^{-3}$ F, and $E = 110 \sin 415t$ V (66 cycles/sec).

24. Find a general solution of the homogeneous linear ODE corresponding to the ODE in Prob. 23.

25. Find the steady-state current in the *RLC*-circuit in Fig. 71 when $R = 50\ \Omega$, $L = 30$ H, $C = 0.025$ F, $E = 200 \sin 4t$ V.

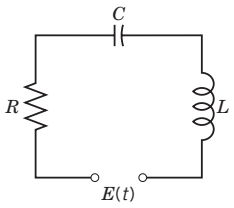


Fig. 71. *RLC*-circuit

26. Find the current in the *RLC*-circuit in Fig. 71 when $R = 40\ \Omega$, $L = 0.4$ H, $C = 10^{-4}$ F, $E = 220 \sin 314t$ V (50 cycles/sec).

27. Find an electrical analog of the mass–spring system with mass 4 kg, spring constant 10 kg/sec^2, damping constant 20 kg/sec, and driving force 100 sin 4t nt.

28. Find the motion of the mass–spring system in Fig. 72 with mass 0.125 kg, damping 0, spring constant 1.125 kg/sec^2, and driving force $\cos t - 4 \sin t$ nt, assuming zero initial displacement and velocity. For what frequency of the driving force would you get resonance?

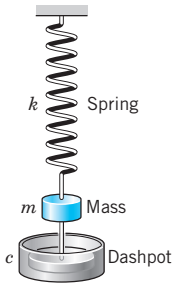


Fig. 72. Mass–spring system

29. Show that the system in Fig. 72 with $m = 4$, $c = 0$, $k = 36$, and driving force $61 \cos 3.1t$ exhibits beats. *Hint:* Choose zero initial conditions.

30. In Fig. 72, let $m = 1$ kg, $c = 4$ kg/sec, $k = 24$ kg/sec^2, and $r(t) = 10 \cos \omega t$ nt. Determine w such that you get the steady-state vibration of maximum possible amplitude. Determine this amplitude. Then find the general solution with this ω and check whether the results are in agreement.

SUMMARY OF CHAPTER 2
Second-Order Linear ODEs

Second-order linear ODEs are particularly important in applications, for instance, in mechanics (Secs. 2.4, 2.8) and electrical engineering (Sec. 2.9). A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{(Sec. 2.1).} \tag{1}$$

(If the first term is, say, $f(x)y''$, divide by $f(x)$ to get the "**standard form**" (1) with y'' as the first term.) Equation (1) is called **homogeneous** if $r(x)$ is zero for all x considered, usually in some open interval; this is written $r(x) \equiv 0$. Then

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Equation (1) is called **nonhomogeneous** if $r(x) \not\equiv 0$ (meaning $r(x)$ is not zero for some x considered).

For the homogeneous ODE (2) we have the important **superposition principle** (Sec. 2.1) that a linear combination $y = ky_1 + ly_2$ of two solutions y_1, y_2 is again a solution.

Two *linearly independent* solutions y_1, y_2 of (2) on an open interval I form a **basis** (or **fundamental system**) of solutions on I. and $y = c_1y_1 + c_2y_2$ with arbitrary constants c_1, c_2 a **general solution** of (2) on I. From it we obtain a **particular solution** if we specify numeric values (numbers) for c_1 and c_2, usually by prescribing two **initial conditions**

(3) $y(x_0) = K_0, \qquad y'(x_0) = K_1 \qquad (x_0, K_0, K_1$ given numbers; Sec. 2.1).

(2) and (3) together form an **initial value problem**. Similarly for (1) and (3).

For a nonhomogeneous ODE (1) a **general solution** is of the form

(4) $$y = y_h + y_p \qquad \text{(Sec. 2.7)}.$$

Here y_h is a general solution of (2) and y_p is a particular solution of (1). Such a y_p can be determined by a general method (***variation of parameters***, Sec. 2.10) or in many practical cases by the ***method of undetermined coefficients***. The latter applies when (1) has constant coefficients p and q, and $r(x)$ is a power of x, sine, cosine, etc. (Sec. 2.7). Then we write (1) as

(5) $$y'' + ay' + by = r(x) \qquad \text{(Sec. 2.7)}.$$

The corresponding homogeneous ODE $y' + ay' + by = 0$ has solutions $y = e^{\lambda x}$, where λ is a root of

(6) $$\lambda^2 + a\lambda + b = 0.$$

Hence there are three cases (Sec. 2.2):

Case	Type of Roots	General Solution
I	Distinct real λ_1, λ_2	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
II	Double $-\frac{1}{2}a$	$y = (c_1 + c_2x)e^{-ax/2}$
III	Complex $-\frac{1}{2}a \pm i\omega^*$	$y = e^{-ax/2}(A \cos \omega^* x + B \sin \omega^* x)$

Here ω^* is used since ω is needed in driving forces.

Important applications of (5) in mechanical and electrical engineering in connection with ***vibrations*** and ***resonance*** are discussed in Secs. 2.4, 2.7, and 2.8.

Another large class of ODEs solvable "algebraically" consists of the **Euler–Cauchy equations**

(7) $$x^2y'' + axy' + by = 0 \qquad \text{(Sec. 2.5)}.$$

These have solutions of the form $y = x^m$, where m is a solution of the auxiliary equation

(8) $$m^2 + (a - 1)m + b = 0.$$

Existence and uniqueness of solutions of (1) and (2) is discussed in Secs. 2.6 and 2.7, and ***reduction of order*** in Sec. 2.1.